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ANALYTIC AND COANALYTIC FAMILIES OF ALMOST DISJOINT FUNCTIONS

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Abstract. If $\mathcal{F} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is an analytic family of pairwise eventually different functions then the following strong maximality condition fails: For any countable $\mathcal{H} \subseteq {}^{\mathbb{N}}\mathbb{N}$, no member of which is covered by finitely many functions from $\mathcal{F}$, there is $f \in \mathcal{F}$ such that for all $h \in \mathcal{H}$ there are infinitely many integers k such that $f(k) = h(k)$. However if $V = L$ then there exists a coanalytic family of pairwise eventually different functions satisfying this strong maximality condition.

§1. Introduction. It is a well known phenomenon of descriptive set theory that subsets of the reals requiring the axiom of choice in order to exist do not have nice descriptions. For example:

- (Suslin [5]) No well ordering of an uncountable set of reals is analytic.
- (Sierpinski) No ultrafilter is measurable or has the property of Baire.
- (Talagrand [11]) The intersection of countably many nonmeasurable filters is nonmeasurable.
- (Mathias [7]) There is no analytic maximal almost disjoint family.

Since many variations on the theme of a maximal almost disjoint family have been explored, the last fact raises a series of questions about the definability properties of other such maximal families. It is the purpose of this paper to analyze one instance of this question for the case of almost disjoint families obtained from graphs. The following definition clarifies this. A family of functions $\mathcal{F} \subseteq {}^{\mathbb{N}}\mathbb{N}$ will be said to be *eventually different* if for any two f and g in $\mathcal{F}$ there is some k such that $f(n) \neq g(n)$ for $n \geq k$. A maximal eventually different family is one which is maximal with respect to this property. The following question remains open:

Question 1.1. Is there an analytic (or even closed) maximal, eventually different family?

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However, it will be shown that the σ-version of the question can be answered satisfactorily.

DEFINITION 1.2. If $\mathcal{F} \subseteq {}^{\mathbb{N}}\mathbb{N}$ and $h \in {}^{\mathbb{N}}\mathbb{N}$ then define h to be finitely covered by $\mathcal{F}$ if there is a finite subset $\mathcal{C} \subseteq \mathcal{F}$ such that $h(k) \in \{f(k)\}_{f\in\mathcal{C}}$ for all but finitely many integers k.

DEFINITION 1.3. An eventually different family of functions $\mathcal{F} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is strongly maximal iff for any countable $\mathcal{H} \subseteq {}^{\mathbb{N}}\mathbb{N}$, no member of which is finitely covered by $\mathcal{F}$, there is $f \in \mathcal{F}$ such that for all $h \in \mathcal{H}$ there are infinitely many integers k such that $f(k) = h(k)$.

THEOREM 1.4. *There is no analytic, strongly maximal, eventually different family.*

As already mentioned, this result should be viewed as an answer to the $\sigma-$ variant of Question 1.1. The σ-variants of various cardinal invariants have been investigated by Brendle and others, [1] and [4]. For example, Solecki has characterized the analytic P-ideals as those very simply induced from a sequence of lower semicontinuous submeasures, [10]. As another illustrative example, it is worth quoting the following result which is similar in spirit to Theorem 1.4.

THEOREM 1.5 (Todorčević, [12]). *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal families of subsets of $\mathbb{N}$ (in other words, if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $|A \cap B| < \aleph_0$) and $\mathcal{A}$ is analytic. Then $\mathcal{A}$ is covered by a countable family orthogonal to $\mathcal{B}$ if and only if every countable subset of $\mathcal{B}$ can be separated from $\mathcal{A}$.*

Note that a Hausdorff gap provides a counterexample to Theorem 1.5 if the hypothesis on analyticity is dropped. The σ-variant hypothesis that every countable subset of $\mathcal{B}$ can be separated from $\mathcal{A}$ is essential here. Whether this is also the case for Theorem 1.4 remains to be seen.

As with the other variations on the theme of maximal almost disjoint family Theorem 1.4 has a companion theorem for which we provide a fully detailed proof in section 3.

THEOREM 1.6. *The axiom of constructibility implies the existence of a coanalytic strongly maximal, eventually different family.*

The theorems 1.4 and 1.6 together completely answer the question of possible complexities of strongly mad families of functions.

§2. Strongly Maximal Almost Disjoint Families can not be Analytic. The purpose of this section is to prove the following theorem.

THEOREM 2.1. *There is no analytic, strongly maximal, almost disjoint family in ${}^{\mathbb{N}}\mathbb{N}$.*

We assume towards a contradiction that there exists an analytic, strongly maximal, almost disjoint family of functions $\mathcal{F}$. Since it is analytic there exists a closed subset of the irrationals T and a continuous function $\Phi : T \to {}^{\mathbb{N}}\mathbb{N}$ whose range is $\mathcal{F}$. Using Φ we define a stratification T_α of the family $\mathcal{F}$ (Lemma 2.4). From this stratification we get, using Lemma 2.6, a countable family of functions that allow us to derive a contradiction using the strong maximality condition.

Lemma 2.2. *If $I \subseteq {}^{\mathbb{N}}\mathbb{N}$ is an infinite family of pairwise eventually different functions then*

$$\lim_{k\to\infty} |\{t(k) : t \in I\}| = \infty.$$

Proof. Suppose that I provides a counterexample. Then there exist an infinite set $K \subseteq \mathbb{N}$ and an integer m such that $|\{t(k) : t \in I\}| = m$ for each $k \in K$. For each $k \in K$ choose $a_i^k \in \mathbb{N}$, $i < m$, such that $\{t(k) \mid t \in I\} = \{a_0^k, a_1^k, \dots, a_{m-1}^k\}$. Let F be a free ultrafilter on K and choose distinct t_0, t_1, ... ,t_m in I. For each $i < m$ let n_i be such that $0 \le n_i < m$ and $S_i = \{k \in K : t_i(k) = a_{n_i}^k\} \in F$. Then there are i and j such that $0 \le i < j \le m$ and $n_i = n_j$. Then however the functions t_i and t_j agree on the infinite set $S_i \cap S_j$ contradicting that they are eventually different. ⊣

Definition 2.3. If T is a closed subset of the irrationals, $\Phi : T \to {}^{\mathbb{N}}\mathbb{N}$ is a continuous function and $U \subseteq T$ is a relatively open set define

$$B_k(T, \Phi, U) = \mathbb{N} \setminus \{\Phi(t)(k)\}_{t\in U}.$$

Define

$$T' = T \setminus \bigcup \left\{ U \subseteq T : U \text{ is open and } \lim_{k\to\infty} |B_k(T, \Phi, U)| = \infty \right\}.$$

Let $T_0 = T$, $T_{\alpha+1} = (T_\alpha)'$ and, if β is a limit, then $T_\beta = \bigcap_{\alpha\in\beta} T_\alpha$. Let T_∞ be equal to any (and all) T_α such that $T_{\alpha+1} = T_\alpha$.

Lemma 2.4. *If $T \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a closed subset of the irrationals and $\Phi : T \to {}^{\mathbb{N}}\mathbb{N}$ is a continuous function whose range is a family of pairwise eventually different functions then $T_\infty = \emptyset$. Moreover the least α such that $T_\alpha = T_\infty = \emptyset$ is countable.*

Proof. First note that if $U \subseteq T_\infty$ is non-empty then there is some integer j and infinitely many integers k such that $|B_k(T_\infty, \Phi, U)| < j$. In particular, there is some k such that $\{\Phi(t)(k)\}_{t\in T_\infty}$ is infinite and so, since Φ is continuous, it is possible to find non-empty open sets U and V in T_∞ on which Φ has disjoint ranges.

Now construct open sets U_i and V_i as well as integers m_i such that:

- $U_0 = U$ and $V_0 = V$
- $U_{i+1} \subseteq \overline{U_i}$ and $V_{i+1} \subseteq \overline{V_i}$
- $\Phi(v)(m_i) = \Phi(u)(m_i)$ for all $v \in \overline{V_i}$ and $u \in \overline{U_i}$
- $m_i < m_{i+1}$
- the diameters of V_i and U_i are both less than $1/i$.

If this can be accomplished then choosing $v \in \bigcap_{i=0}^{\infty} \overline{V_i}$ and $u \in \bigcap_{i=0}^{\infty} \overline{U_i}$ (note that there is no real choice involved, by construction these intersections are singletons) yields that $\Phi(u)$ and $\Phi(v)$ agree on each m_i, But since Φ has disjoint ranges on U and V it follows that $\Phi(u) \ne \Phi(v)$ and so they are eventually different.

To show that the inductive construction can be carried out, suppose that U_i, V_i and m_i are given. Let $U_i' \subseteq \overline{U_i}$ and $V_i' \subseteq \overline{V_i}$ be open sets of diameter less than $1/(i+1)$. It has already been noted that there is some integer j and infinitely many integers k such that $|B_k(T_\infty, \Phi, U_i')| < j$. Moreover the definition of T_∞ guarantees that the range of Φ on V_i' is infinite. Therefore, by Lemma 2.2,

$$\lim_{k\to\infty} |\{\Phi(t)(k) : t \in V_i'\}| = \infty.$$

It is therefore possible to choose $m_{i+1} > m_i$ such that

$$|\,\{\Phi(t)(m_{i+1}) : t \in V_i'\}\,| > |B_{m_{i+1}}(T_\infty, \Phi, U_i')|.$$

Choose

$$n \in \{\Phi(t)(m_{i+1}) : t \in V_i'\} \setminus B_{m_{i+1}}(T_\infty, \Phi, U_i')$$

and let $U_{i+1} = \{u \in U_i' : \Phi(u)(m_{i+1}) = n\}$ and $V_{i+1} = \{v \in V_i' : \Phi(v)(m_{i+1}) = n\}$. These are both non-empty by the choice of n and it is immediate that the induction hypotheses are satisfied.

The second claim is immediate as there are only countably many basic open sets in $\mathbb{R}$. ⊣

Lemma 2.5. *For any integer k and for any family $\mathcal{F}$ of pairwise disjoint functions from $k+1$ to $\mathbb{N}$ and any mapping*

$$\Psi : \prod_{n=0}^{k} n+1 \to [\mathcal{F}]^k$$

there is some $g \in \prod_{n=0}^{k} n+1$ such that $g \not\subseteq \cup\Psi(g)$.

Proof. Suppose that Ψ is a counterexample. Since $g(0) = 0$ for each $g \in \prod_{n=0}^{k} n+1$ it follows that there is a unique member f in $\mathcal{F}$ such that $f(0) = 0$ and, furthermore, $f \in \Psi(g)$ for each $g \in \prod_{n=0}^{k} n+1$. Let $f_0 = f$ and define $\bar{g}_0(0) = 0$. Suppose now that $f_i \in \mathcal{F}$ have been defined for $i \le m < k$ and that $\bar{g}_m : m+1 \to \mathbb{N}$ is also defined so that:

- $f_i \ne f_j$ unless $i = j$
- $\bar{g}_m(i) \le i$ for each $i \le m$
- $\bar{g}_m \supseteq \bar{g}_i$ for $i < m$
- if $g \in \prod_{n=0}^{k} n+1$ and $\bar{g}_m \subseteq g$ then $\{f_i\}_{i\le m} \subseteq \Psi(g)$.

Let $\bar{g}_{m+1}(m+1) \le m+1$ be such that $\bar{g}_{m+1}(m+1) \notin \{f_i(m+1)\}_{i\le m}$ and $\bar{g}_{m+1} \supseteq \bar{g}_m$. Let $f_{m+1} \in \mathcal{F}$ be the unique member f of $\mathcal{F}$ such that $\bar{g}_{m+1}(m+1) = f(m+1)$ and note that if $g \in \prod_{n=0}^{k} n+1$ and $\bar{g}_{m+1} \subseteq g$ then $\{f_i\}_{i\le m+1} \subseteq \Psi(g)$. Then $\bar{g}_k \in \prod_{n=0}^{k} n+1$ and $\{f_i\}_{i\le k} \subseteq \Psi(\bar{g}_k)$ contradicting the assumption that $|\Psi(\bar{g}_k)| = k$. ⊣

Lemma 2.6. *If $\mathcal{F} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is an analytic family of pairwise eventually different functions and $\{B_n\}_{n=0}^{\infty}$ is a sequence of non-empty finite subsets of $\mathbb{N}$ such that $\lim_{n\to\infty} |B_n| = \infty$ then*

$$\left\{ b \in \prod_{n=0}^{\infty} B_n : b \textit{ is not finitely covered by } \mathcal{F} \right\}$$

is comeagre, where $\prod_{n=0}^{\infty} B_n$ has the product topology with each B_n being discrete.

PROOF. To begin, let

$$\mathcal{C}_k = \left\{ b \in \prod_{n=0}^{\infty} B_n : (\exists \mathcal{H} \in [\mathcal{F}]^k) b \subseteq^* \cup \mathcal{H} \right\}$$

and note that $\mathcal{C}_k$ is Σ^1_1. If each $\mathcal{C}_k$ is meagre then so is $\bigcup_{k=1}^{\infty} \mathcal{C}_k$. Hence toward a contradiction it may be assumed that there is some k such that $\mathcal{C}_k$ is not meagre. Since $\mathcal{C}_k$ is invariant under finite modifications and it has the Property of Baire, it follows that $\mathcal{C}_k$ is co-meagre.

Let R be the relation on $\prod_{n=0}^{\infty} B_n \times \mathcal{F}^k$ defined by $R(b, (f_1, f_2, \dots, f_k))$ holds if and only if $b \subseteq^* f_1 \cup f_2 \cup \dots \cup f_k$. Then R is analytic and by the Jankov – von Neumann Uniformization Theorem there is a Baire measurable function $\Phi : \mathcal{C}_k \to \mathcal{F}^k$ such that $R(b, \Phi(b))$ holds for all $b \in \mathcal{C}_k$. Let $\mathcal{C} \subseteq \mathcal{C}_k$ be a dense G_δ such that $\Phi \restriction \mathcal{C}$ is continuous. Let $\Phi(b) = (\Phi_1(b), \Phi_2(b), \dots, \Phi_k(b))$.

NOTATION 2.7. Let $\prod_{n=0}^{<\infty} B_n$ denote $\bigcup_{u=1}^{\infty} \prod_{n=0}^{u} B_n$ and for $s \in \prod_{n=0}^{<\infty} B_n$ let $[s] = \{b \in \prod_{n=0}^{\infty} B_n : s \subseteq b\}$.

Let

$$\mathcal{D}'_m = \left\{ b \in \prod_{n=0}^{\infty} B_n : (\forall 1 \le i < j \le k)(\forall n > m) \;\; \Phi_i(b)(n) \ne \Phi_j(b)(n) \right\}$$

and let $\mathcal{D}_m = \{b \in \mathcal{D}'_m : (\forall n > m)(\exists j \le k) \;\; b(n) = \Phi_j(b)(n)\}$. Since $\mathcal{C}$ is a subset of $\bigcup_{m=1}^{\infty} \mathcal{D}_m$ there is some m such that $\mathcal{C} \cap \mathcal{D}_m$ is not meagre. Since this set is Borel, there is some $s \in \prod_{n=0}^{<\infty} B_n$ such that $\mathcal{D}_m \cap \mathcal{C}$ is co-meagre in the open set $[s]$. Before continuing, the following claim will be established:

CLAIM 2.8. Given $s_1 \in \prod_{n=0}^{<\infty} B_n$ and $s_2 \in \prod_{n=0}^{<\infty} B_n$ and $1 \le a \le b \le k$ there are $t_1 \supseteq s_1$ and $t_2 \supseteq s_2$ and there is a comeagre set $\mathcal{E} \subseteq [t_1] \times [t_2]$ such that one of the following two alternatives holds:

$$(\exists v)(\forall (e_1, e_2) \in \mathcal{E})(\forall n > v) \;\; \Phi_a(e_1)(n) \ne \Phi_b(e_2)(n) \tag{2.2.1}$$

$$(\forall (e_1, e_2) \in [s_1] \times [s_2]) \;\; \Phi_a(e_1) = \Phi_b(e_2) \tag{2.2.2}$$

PROOF. Suppose that alternative 2.2.2 fails. Choose $\sigma_1 \in [s_1]$ and $\sigma_2 \in [s_2]$ such that $\Phi_a(\sigma_1) \ne \Phi_b(\sigma_2)$ and, using the continuity of Φ, choose j such that $\Phi_a([\sigma_1 \restriction j]) \cap \Phi_b([\sigma_2 \restriction j]) = \emptyset$. It follows that if

$$\mathcal{E}_\ell = \{(e_1, e_2) \in [\sigma_1 \restriction j] \times [\sigma_2 \restriction j] : (\forall n > \ell) \;\; \Phi_a(e_1)(n) \ne \Phi_b(e_2)(n)\}$$

then $([\sigma_1 \restriction j] \times [\sigma_2 \restriction j]) \cap (\mathcal{C} \times \mathcal{C}) = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ and so there are some integer v, $t_1 \supseteq \sigma_1 \restriction j$, and $t_2 \supseteq \sigma_2 \restriction j$ such that $\mathcal{E}_v$ is co-meagre in $[t_1] \times [t_2]$. Let $\mathcal{E} = \mathcal{E}_v$. ⊣

Now let $\{(i_n, j_n, a_n, b_n)\}_{n \in L}$ enumerate all quadruples (i, j, a, b) such that $\{i, j\} \in [(k+1)!]^2$ and $1 \le a < b \le k$. Choose $\{s_i\}_{i=1}^{(k+1)!} \subseteq \prod_{i=0}^{<\infty} B_n$ such that the sets $\{[s_i]\}_{i=1}^{(k+1)!}$ are pairwise disjoint and $s \subseteq \bigcap_{i=1}^{(k+1)!} s_i$. Let $A = \prod_{i=1}^{(k+1)!} [s_i] \cap \prod_{i=1}^{(k+1)!} \mathcal{C} \cap \prod_{i=1}^{(k+1)!} \mathcal{D}_m$. Then construct by induction sets $\{A_n\}_{n \le L}$, integers M_n and $\{\{s_i^n\}_{i=1}^{(k+1)!}\}_{n \le L}$ satisfying the following:

- $A_0 = A$,

- $s_i^0 = s_i$ for each i with $1 \leq i \leq (k+1)!$,
- A_n is comeagre in $\prod_{i=1}^{(k+1)!}[s_i^n]$,
- $s_i^n \subseteq s_i^{n+1}$ for each $i \in (k+1)!$,
- if there is a pair $(\bar{e}_1, \bar{e}_2) \in [s_{i_{n-1}}^{n-1}] \times [s_{j_{n-1}}^{n-1}]$ such that $\Phi_{a_{n-1}}(\bar{e}_1) \neq \Phi_{b_{n-1}}(\bar{e}_2)$ then for each $(e_1, e_2, \ldots, e_{(k+1)!}) \in A_n$ and for all $z > M_n$ the inequality $\Phi_{a_{n-1}}(e_{i_{n-1}})(z) \neq \Phi_{b_{n-1}}(e_{j_{n-1}})(z)$ holds.

The construction is easily carried out using the Claim at each stage. To be precise, given A_n and $\{s_i^n\}_{i=1}^{k!}$ if there is no pair $(e_1, e_2) \in [s_{i_n}^n] \times [s_{j_n}^n]$ such that $\Phi_{a_n}(e_1) \neq \Phi_{b_n}(e_2)$ then let $\{s_i^{n+1}\}_{i=1}^{k!} = \{s_i^n\}_{i=1}^{k!}$ and $A_{n+1} = A_n$ and M_{n+1} is irrelevant in this case. Otherwise, use the Claim to find $t_1 \supseteq s_{i_n}^n$ and $t_2 \supseteq s_{j_n}^n$ and a comeagre set $\mathcal{E} \subseteq [t_1] \times [t_2]$ such that alternative 2.2.1 holds and this is witnessed by v. Then let $\{s_i^{n+1}\}_{i=1}^{k!} = \{s_i^n\}_{i=1}^{k!}$ for $i \notin \{i_n, j_n\}$ and let $s_{i_n}^{n+1} = t_1$ and $s_{j_n}^{n+1} = t_2$, let

$$A_{n+1} = \left(A_n \cap \prod_{i=1}^{k!}[s_i^{n+1}]\right) \cap \pi^{-1}\mathcal{E}$$

where π is the projection of $({}^{\mathbb{N}}\mathbb{N})^{k!}$ onto $({}^{\mathbb{N}}\mathbb{N})^{\{i_n, j_n\}}$. Let $M_{n+1} = v$.

Now let $J \in \mathbb{N}$ be such that $J > m$, $J \geq \max\{M_i\}_{i \leq L}$, $J \geq \max\{|s_i^L|\}_{i=0}^{(k+1)!}$, and $|B_u| > k$ for each $u \geq J$. From the last clause it is possible to find $\{x_q^i\}_{q \in i} \in [B_{J+i}]^i$, for $1 \leq i \leq k+1$. Choose a bijection $\beta : \mathbb{N}^{k+1} \to \mathbb{N}^{k+1}$ such that for $b \in \mathbb{N}^{k+1}$ if $b(i) \in i+1$ we have $\beta(b)(i) = x_{b(i)}^{i+1}$. The use of β in the below is essentially to conjugate $\prod_{n=0}^{k}\{x_q^{n+1} \mid 0 \leq q \leq n\}$ onto $\prod_{n=0}^{k} n+1$. Let $\{\theta_u\}_{u=1}^{(k+1)!}$ enumerate $\prod_{n=0}^{k} n+1$. Choose $w_i \supseteq s_i^L$ so that $|w_i| = J$ for each $i \leq (k+1)!$ and, then, let $w_u^* = w_u{}^\frown\beta(\theta_u)$. Since A_L is co-meagre in $\prod_{i=1}^{(k+1)!}[s_i^L]$ it follows that it is possible to choose $(\zeta_1, \zeta_2, \ldots, \zeta_{(k+1)!}) \in A_L \cap \prod_{i=1}^{(k+1)!}[w_i^*]$. Now let $h_{i,a} \in \mathbb{N}^{k+1}$ be defined by setting $h_{i,a}(n) = \Phi_a(\zeta_i)(J+n)$ and let $\mathcal{H}$ be the set of all $\beta^{-1}(h_{i,a})$ and note that this forms a family of disjoint functions. Then define $\Psi : \prod_{n=0}^{k} n+1 \to \mathcal{H}^k$ by setting $\Psi(\theta_i) = \{\beta^{-1}(h_{i,a})\}_{a=1}^{k}$. The definition of $\mathcal{D}_m$ and the fact that $J > m$ guarantees that $\Psi(b) \supseteq b$ for every $b \in \prod_{n=0}^{k} n+1$. This contradicts Lemma 2.5. $\dashv$

Fix a countable base $\mathcal{B}$ for T. Let $\alpha \in \omega_1$ be such that $\emptyset = T_\alpha$. For each $U \in \mathcal{B}$ and each $\beta \in \alpha$ such that $U \cap T_\beta \neq \emptyset$ and $\lim_{k\to\infty} |B_k(T_\beta, \Phi, U \cap T_\beta)| = \infty$ use Lemma 2.6 to find a function $h(\beta, U)$ which is not finitely covered by $\mathcal{F}$ such that if $B_k(T_\beta, \Phi, U \cap T_\beta) \neq \emptyset$ then $h(\beta, U)(k) \in B_k(T_\beta, \Phi, U \cap T_\beta)$. Now use the strong maximality of $\mathcal{F}$ to find $t \in T$ such that $\Phi(t)(n) = h(\beta, U)(n)$ for infinitely many n for every relevant β and U. Let γ be the greatest ordinal such that $t \in T_\gamma$ and let $U \in \mathcal{B}$ be such that $t \in U$ and $\lim_{k\to\infty} |B_k(T_\gamma, \Phi, U \cap T_\gamma)| = \infty$. Choose an integer k such that $\Phi(t)(k) = h(\gamma, U)(k) \in B_k(T_\gamma, \Phi, U \cap T_\gamma)$. Since $t \in T_\gamma \cap U$ it follows that $\Phi(t)(k) \notin B_k(T_\gamma, \Phi, U \cap T_\gamma)$. This contradiction establishes the main theorem.

§3. Very Mad Families Can be Coanalytic. In this section we prove the following theorem.

Theorem 3.1. *The Axiom of Constructibility implies the existence of a* Π^1_1 *strongly maximal, eventually different family.*

The proof is based on the proof of the analogous result for maximal almost disjoint families of subsets of $\mathbb{N}$ by Arnold Miller, see [9]. For background on constructibility see [6, Chap. VI], and [2] in combination with [8] (the theory Basic Set Theory is not strong enough for the use Devlin makes of it, this is analyzed in Mathias paper, and a replacement is offered there that is sufficient for the results we use).

The idea of this proof is that we identify a set of good levels of L (those for which $L_\alpha = \mathrm{Sk}(L_\alpha)$, as defined below). We prove a coding lemma (Lemma 3.2) allowing us to encode these levels into our construction. Then we show that from an encoding of a good level we have access to the limit level after it (Lemma 3.6), which allows us to decide membership (Lemma 3.7).

In this section we choose the sequence coding $\langle \dots \rangle$ and projections π_i to be recursive.

Lemma 3.2. *Let* $A = \{g_n \mid n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ *be an almost disjoint family,* $E \subseteq \mathbb{N} \times \mathbb{N}$ *and* $F = \{f_n \mid n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ *not finitely covered by* A*. Then there exists a function* $g : \mathbb{N} \to \mathbb{N}$ *almost disjoint from all functions in* A*, such that* E *is recursive in* g *and* g *agrees on infinitely many inputs with each member of* F *(*$\forall n \in \mathbb{N}\, |f_n \cap g| = \aleph_0$*).*

Proof. Instead of encoding E directly we encode χ the characteristic function of $\{\langle n, m\rangle \mid (n, m) \in E\}$.

We define g recursively. At step s we extend the initial segment of $\mathbb{N}$ on which g is defined by doing the following:

1. Find $n_{s,i}$, $i \in [0, s]$, such that $n_s < n_{s,0} < n_{s,1} < \cdots < n_{s,s}$, where n_s is the least number where g is not defined yet, and $f_i(n_{s,i})$ is different from all $g_0(n_{s,i}), \dots, g_s(n_{s,i})$. Then define $g(n_{s,i}) = f_i(n_{s,i})$. Also define n_{s+1} to be $n_{s,s} + 1$.
2. Define $g(l)$ for $n_s < l < n_{s+1}$ where g is not yet defined to be the least number different from all $g_0(l), \dots, g_s(l)$.
3. Define $g(n_s)$ to be $\langle k, \langle n_{s+1}, \chi(s)\rangle\rangle$ where k is the least number such that $\langle k, \langle n_{s+1}, \chi(s)\rangle\rangle$ is different from all $g_0(n_s), \dots, g_s(n_s)$. Here the value n_{s+1} is the "pointer" to the next location where a value of χ can be found.

It can now be easily checked that the g constructed satisfies the lemma. $\dashv$

We note that if A, F, E are members of L_α, then g is a member of $L_{\alpha+1}$; the proof shows how to define g from A, F and χ; also E and χ appear at the same level of the constructible hierarchy. Also note that the encoding is uniform: it does not depend on which functions and families we work with. This also means that we can talk about *the* relation encoded in g (later this relation will be the membership relation of a model on $(\mathbb{N}, E)$).

Definition 3.3. For $\alpha > \omega$ we say $L_\alpha = \mathrm{Sk}(L_\alpha)$ iff there exists $\langle h, \varphi, \bar{p}\rangle$ (the witness) such that:

1. h is a Skolem function for all Σ_k formulas for L_α, for some $k \geq 1$,
2. $\bar{p} \in L_\alpha$,

3. $h[\mathbb{N} \times (\mathbb{N} \cup \bar{p})] = L_\alpha$, and
4. $h(n, x) = y \Leftrightarrow L_\alpha \models \varphi(\bar{p}, n, x, y)$.

Lemma 3.4. *The set* $\{\alpha \mid L_\alpha = \mathrm{Sk}(L_\alpha)\}$ *is unbounded in* ω_1.

Proof. First recall from Gödel's proof of CH in L that every constructible real is in L_{ω_1}. From this using the fact all L_β, $\beta < \omega_1$, are countable it follows that the set $\{\beta < \omega_1 \mid \exists r\ [\, r \in L_{\beta+1} \setminus L_\beta \wedge r \in {}^{\mathbb{N}}\mathbb{N}\,]\,\}$ is unbounded in ω_1. So it is sufficient to prove for each β in this set that $L_{\beta+\omega} = \mathrm{Sk}(L_{\beta+\omega})$.

Therefore let r be definable over L_β from a finite sequence of parameters $\bar{q}$, $r = \{\langle m, n\rangle \mid L_\beta \models \psi(m, n, \bar{q})\}$, and such that $r \notin L_\beta$. Then $r \in L_{\beta+\omega}$ so that $L_{\beta+\omega} \models \exists r\ \forall m, n \in \omega\ \big((m, n) \in r \leftrightarrow \psi^{L_\beta}(m, n, \bar{q})\big)$. We can assume that $(\psi, \bar{q})$ is minimal among pairs of formulas and parameters that define a new real over L_β. This means that $(\psi, \bar{q})$ is definable from L_β, say by formula φ.

Let $h : \mathbb{N} \times L_{\beta+\omega} \to L_{\beta+\omega}$ be a definable Skolem function for Σ_{k+2} formulas with $k \in \mathbb{N}$ such that $\psi, \varphi \in \Sigma_k$.

Let $X = h[\mathbb{N} \times (\mathbb{N} \cup \{L_\beta\})]$ and note that $\bar{q} \in X$. Then since $X \prec_{k+2} L_{\beta+\omega}$ we have that $(X, \in) \models \psi^{L_\beta}(n, m, \bar{q})$ iff $(L_{\beta+\omega}, \in) \models \psi^{L_\beta}(n, m, \bar{q})$ and $(X, \in) \models \exists r\ \forall m, n \in \omega\ \big((m, n) \in r \leftrightarrow \psi^{L_\beta}(m, n, \bar{q})\big)$, which shows r is in $(X, \in)$. Also since for every Σ_{k+2} formula there is an equivalent formula such that if there is a witness for the existential quantifier there is a unique witness for the existential quantifier, every element of X is definable from L_β.

By the condensation lemma [2, Theorem II.5.2] we have a π such that $\pi : (X, \in) \cong (L_\alpha, \in)$, $\alpha \leq \beta + \omega$ and α is a limit ordinal; this π is the identity on transitive sets, in particular on the natural numbers. From this we get $(X, \in) \models \psi^{L_\beta}(n, m, \bar{q})$ iff $(L_\alpha, \in) \models \psi^{\pi L_\beta}(n, m, \pi\bar{q})$ and $(L_\alpha, \in) \models \exists r\ \forall m, n \in \omega\big((m, n) \in r \leftrightarrow \psi^{\pi L_\beta}(m, n, \pi\bar{q})\big)$, which shows that r is in L_α. So since $\alpha \leq \beta + \omega$, $r \notin L_\beta$, $r \in L_\alpha$, and α is a limit ordinal, we get $\alpha = \beta + \omega$. This means $X \cong L_{\beta+\omega}$. Now since L_β is the level after which r appears, also in X, $\pi(L_\beta)$ maps to L_β under this isomorphism. But everything in X is Σ_{k+2} definable from L_β. This implies that everything in $L_{\beta+\omega}$ is Σ_{k+2} definable from L_β. Now note that X is the image of $\mathbb{N} \times (\mathbb{N} \cup \{L_\beta\})$ by a Σ_{k+2} Skolem function, which with the fact that everything in $L_{\beta+\omega}$ is Σ_{k+2} definable from L_β implies that $X = L_{\beta+\omega}$, as was to be shown. ⊣

Enumerate the set $\{\alpha < \omega_1 \mid L_\alpha = \mathrm{Sk}(L_\alpha)$ in increasing order by $\langle \beta_\gamma \mid \gamma < \omega_1\rangle$. Note that by absoluteness of the notion $L_\alpha = \mathrm{Sk}(L_\alpha)$ and the fact that limit levels of the constructible hierarchy are closed under certain simple recursions, we have that $L_{\beta_\gamma+\omega} \models$"$\langle \beta_{\gamma'} \mid \gamma' \leq \gamma\rangle$ is an initial segment of the increasing enumeration of ordinals α such that $L_\alpha = \mathrm{Sk}(L_\alpha)$".

Lemma 3.5. *If* $L_\alpha = \mathrm{Sk}(L_\alpha)$*, then there is an* $E \subseteq \mathbb{N} \times \mathbb{N}$ *such that* $E \in L_{\alpha+\omega}$ *and* $(L_\alpha, \in) \cong (\mathbb{N}, E)$.

Proof. Let $L_\alpha = \mathrm{Sk}(L_\alpha)$ be witnessed by $\langle h, \varphi, \bar{p}\rangle$. We will show in two steps that then there is an E as in the statement of the lemma. We show first that $h \in L_{\beta+\omega}$, and then we show how to construct E from h.

Note that $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle) \in L_{\alpha+\omega}$: we follow the ideas from pages 40 and 41 of [2]. The lemma Devlin proves there is not correct, see [8], but the method can be used here. We have a function f such that $f(0)$ is the set of all primitive formulas of set theory, and $f(i+1)$ is the set of all formulas formed from the formulas in $f(i)$ by conjunction, disjunction, implication, and quantification. Then we construct a function g such that $g(i)$ is a set of pairs, first coordinate a formula φ from $f(i)$, second coordinate a sequence $\bar{x}$ of elements of L_α such that $\varphi(\bar{x})$ is true in $(L_\alpha, \in, \bar{p})$. All these elements are in $L_{\alpha+n}$ for some n. Then in $L_{\alpha+n+1}$ we can construct all $g \restriction k$ for $k \in \mathbb{N}$. So in $L_{\alpha+n+2}$ we can use the recursive definition of g to construct it. From g we get $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$ as the subset of the image consisting of all formulas with no free variables. (Note that this, and the following, are all uniform with respect to the sequence $\bar{p}$, but, for notational convenience, we'll leave it implicit as a parameter.)

Let $e : \mathbb{N} \to \mathbb{N} \times (\mathbb{N} \cup \bar{p})$ be the definable bijection

$$e(n) = \begin{cases} (\pi_0(n), \bar{p}_{\pi_1(n)}), & \text{if } \pi_1(n) < \mathrm{lh}(\bar{p}); \\ (\pi_0(n), \pi_1(n) - \mathrm{lh}(\bar{p})), & \text{otherwise,} \end{cases}$$

and φ_e the formula defining e, i.e. $\varphi_e(n, x, y) \Leftrightarrow e(n) = (x, y)$ (this formula defines e in any $L_{\alpha+4}$ with $\alpha > \omega$ and $\bar{p} \in L_\alpha$ and is absolute for these levels).

Define $\tilde{e} : \mathbb{N} \to \mathbb{N}$ from this by setting $\tilde{e}(0) = 0$ and $\tilde{e}(n+1) = k$ where k is the least number bigger than $\tilde{e}(n)$ such that $\ulcorner\psi(k, \tilde{e}(n))\urcorner \in \mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$, where $\psi(k, \tilde{e}(n))$ is the formula

$$\forall l \le \tilde{e}(n)\ \forall y_0, y_1 \ \big[\ \varphi(\bar{p}, \pi_0(e(l)), \pi_1(e(l)), y_0) \ \wedge\ \varphi(\bar{p}, \pi_0(e(k)), \pi_1(e(k)), y_1) \to \\ \big(\exists z\ (z \in y_0 \wedge z \notin y_1) \ \vee\ (z \notin y_0 \wedge z \in y_1)\big)\ \big],$$

in which φ is the formula defining h, and which after elimination of e in favor of its definition becomes

$$\forall l \le \tilde{e}(n)\ \forall l_0, l_1, k_0, k_1\ \big\{\ \varphi_e(l, l_0, l_1)\ \wedge\ \varphi_e(k, k_0, k_1) \to \\ \forall y_0, y_1 \big[\ \varphi(\bar{p}, l_0, l_1, y_0)\ \wedge\ \varphi(\bar{p}, k_0, k_1, y_1) \to \\ \big(\ \exists z\ (z \in y_0 \wedge z \notin y_1)\ \vee\ (z \notin y_0 \wedge z \in y_1)\ \big)\ \big]\ \big\}.$$

Note $\psi(k, \tilde{e}(n))$ is the formula expressing $\forall l \le \tilde{e}(n)\ h(e(k)) \ne h(e(l))$, and a Gödel number for $\psi(k, \tilde{e}(n))$ can be recursively obtained from k and n (the function $(k, m) \mapsto \ulcorner\forall x \theta_m(x) \to \psi(k, x)\urcorner$ (where $\theta_m(x)$ is the formula defining the natural number m) is in $L_{\omega+\omega}$, but $\tilde{e}$ which is recursively defined from it and $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$ can be constructed at the level of L after $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$ is constructed).

Let $\varphi_{\tilde{e}}(n, m)$ be such that $\varphi_{\tilde{e}}(n, m) \Leftrightarrow \tilde{e}(n) = m$.

These definitions have been made so that $h \circ e \circ \tilde{e} : \mathbb{N} \to L_\alpha$ is an enumeration of $h[\mathbb{N} \times (\mathbb{N} \cup \bar{p})]$ without repetitions. We will set up the model $(\mathbb{N}, E)$ such that the number $m \in \mathbb{N}$ will represent the set $h(e(\tilde{e}(m)))$. It is then clear that we want $n\ E\ m$ iff $h(e(\tilde{e}(n))) \in h(e(\tilde{e}(m)))$.

We show $E \in L_{\alpha+k}$ for some $k \in \mathbb{N}$ by eliminating all functions in favor of their definitions in the statement $h(e(\tilde{e}(n))) \in h(e(\tilde{e}(m)))$, and then noting this statement is true of (n, m) iff the Gödel number of the formula resulting from substituting terms defining n and m in this formula is in $\mathrm{Th}(\langle L_\alpha, E, \bar{p}\rangle)$.

First eliminating h, we get

$$\forall z_n, z_m \left[\left\{ \varphi(\bar{p}, \pi_0(e(\tilde{e}(n))), \pi_1(e(\tilde{e}(n))), z_n) \ \wedge \ \varphi(\bar{p}, \pi_0(e(\tilde{e}(m))), \pi_1(e(\tilde{e}(m))), z_m) \right\} \rightarrow z_n \in z_m \right].$$

Then eliminating e we get

$$\forall x_n, y_n, x_m, y_m \left\{ \ \varphi_e(\tilde{e}(n), x_n, y_n) \ \wedge \ \varphi_e(\tilde{e}(m), x_m, y_m) \rightarrow \forall z_n, z_m \left[\ \varphi(\bar{p}, x_n, y_n, z_n) \ \wedge \ \varphi(\bar{p}, x_m, y_m, z_m) \rightarrow z_n \in z_m \right] \ \right\}.$$

After eliminating $\tilde{e}$ this gives

$$\forall l_n, l_m \left(\ \varphi_{\tilde{e}}(n, l_n) \ \wedge \ \varphi_{\tilde{e}}(m, l_m) \rightarrow \forall x_n, y_n, x_m, y_m \left\{ \ \varphi_e(l_n, x_n, y_n) \ \wedge \ \varphi_e(l_m, x_m, y_m) \rightarrow \forall z_n, z_m \left[\ \varphi(\bar{p}, x_n, y_n, z_n) \ \wedge \ \varphi(\bar{p}, x_m, y_m, z_m) \rightarrow z_n \in z_m \right] \right\} \right).$$

This is a formula in the language $\{\in, \bar{p}\}$ with free variables n and m. The recursive function G that to (n, m) assigns the Gödel number of the formula

$$\forall u, v \ \theta_n(u) \wedge \theta_m(v) \rightarrow \forall l_n, l_m \left(\ \varphi_{\tilde{e}}(u, l_n) \ \wedge \ \varphi_{\tilde{e}}(v, l_m) \rightarrow \forall x_n, y_n, x_m, y_m \left\{ \ \varphi_e(l_n, x_n, y_n) \ \wedge \ \varphi_e(l_m, x_m, y_m) \rightarrow \forall z_n, z_m \left[\ \varphi(\bar{p}, x_n, y_n, z_n) \ \wedge \ \varphi(\bar{p}, x_m, y_m, z_m) \rightarrow z_n \in z_m \right] \right\} \right)$$

is in $L_{\alpha+l}$ for some $l \in \mathbb{N}$ (note: $\varphi_{\tilde{e}}$ uses $\mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$ as a parameter).

This shows that we can define E over $L_{\alpha+l}$ by $(n, m) \in E$ iff $G(n, m) \in \mathrm{Th}(\langle L_\alpha, \in, \bar{p}\rangle)$. $\dashv$

We now define functions (as in [3, page 217]) relating the natural numbers and the real numbers to their representatives in $(\mathbb{N}, E)$.

Define for any $(\mathbb{N}, E) \cong L_\alpha$, $\omega < \alpha < \omega_1$, a recursive function $\mathsf{Nat}_E : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\mathsf{Nat}_E(0) = \text{the unique } u \in \mathbb{N} \text{ such that } \forall l \in \mathbb{N} \ (\neg\, l \ E \ u)$$
$$\mathsf{Nat}_E(n+1) = \text{the unique } u \in \mathbb{N} \text{ such that } \forall l \in \mathbb{N}[(l \ E \ u) \leftrightarrow \big((l \ E \ \mathsf{Nat}_E(n)) \vee (l = \mathsf{Nat}_E(n))\big)].$$

Using this we can define $\mathsf{Real}_E : {}^{\mathbb{N}}\mathbb{N} \rightarrow \mathbb{N}$ a partial function by

$$\mathsf{Real}_E(r) = \text{the unique (if exists) } u \in \mathbb{N} \text{ such that } \forall n, m[r(n) = m \leftrightarrow (\mathbb{N}, E) \models u(\mathsf{Nat}_E(n)) = \mathsf{Nat}_E(m)].$$

Note that $\pi(r) = \mathsf{Real}_E(r)$ for reals r and $\pi(n) = \mathsf{Nat}_E(n)$ for natural numbers.

If $L_\alpha = \mathrm{Sk}(L_\alpha)$, then there exists $\pi : (L_\alpha, \in) \cong (\mathbb{N}, E)$. So the sets $\mathbb{R}_\alpha = {}^{\mathbb{N}}\mathbb{N} \cap L_\alpha$ and $\mathbb{R}_E := \{n \in \mathbb{N} \mid (\mathbb{N}, E) \models n \text{ is a real}\}$ are mapped to each other by the isomorphism. We have in fact that if $r \in \mathbb{R}_\alpha$ then $r(k) = l$ iff $\pi(r)(\mathsf{Nat}_E(k)) = \mathsf{Nat}_E(l)$ is true in $(\mathbb{N}, E)$. So we can define in $L_{\alpha+\omega}$ an enumeration $e_\alpha : \mathbb{N} \rightarrow \mathbb{R}_\alpha$ of all reals in L_α as follows:

First let $e : \mathbb{N} \to \mathbb{R}_E$ be the bijection $e(0) = \min\{\mathbb{R}_E\}$ and $e(n+1) = \min\{m \in \mathbb{R}_E \mid m > e(n)\}$. Then e_α is e composed with the map defined by

$$\{(n,r) \in \mathbb{R}_E \times \mathbb{R}_\alpha \mid \forall k, l \in \mathbb{N}\ (\mathbb{N}, E) \models n(\mathsf{Nat}_E(k)) = \mathsf{Nat}_E(l) \leftrightarrow r(k) = l\} = \{(n,r) \in \mathbb{R}_E \times \mathbb{R}_\alpha \mid \pi(r) = n\}.$$

Now we are ready for the construction of the very mad family $\mathcal{A}$ which we will show is coanalytic. It will be recursively enumerated as $\langle g_\alpha \mid \alpha < \omega_1 \rangle$.

To define g_γ from $\langle g_\alpha \mid \alpha < \gamma \rangle$ we use Lemma 3.2 with $A = A_\gamma = \langle g'_n \mid n \in \mathbb{N} \rangle$, $F = F_\gamma = \langle f_n \mid n \in \mathbb{N} \rangle$ and E as described below.

By Lemma 3.5 we have an E such that $(\mathbb{N}, E) \cong (L_{\beta_\gamma}, \in)$ in $L_{\beta_\gamma + \omega}$.

By induction we will have the set $\{g_\alpha \mid \alpha < \gamma\}$ in $L_{\beta_\gamma+\omega}$ (β_γ as defined on page 8), and by a recursion in $L_{\beta_\gamma+\omega}$ we get the enumeration $\langle g_{\gamma'} \mid \gamma' < \gamma \rangle$ in $L_{\beta_\gamma+\omega}$. We can recursively find an enumeration $\langle g'_n \mid n \in \mathbb{N} \rangle$ of it in $L_{\beta_\gamma+\omega}$ by letting g'_n be the n^{th} member in the enumeration e_{β_γ} of $\mathbb{R}_{\beta_\gamma}$ which is in $\{g_\alpha \mid \alpha < \gamma\}$.

We then recursively define f_n to be the n^{th} member in the enumeration of $\mathbb{R}_{\beta_\gamma}$ which is not finitely covered by $\{g_\alpha \mid \alpha < \gamma\}$. This enumeration will also be in $L_{\beta_\gamma+\omega}$.

After application of Lemma 3.2 (and the observation following it) we get $g_\gamma \in L_{\beta_\gamma+\omega}$. This finishes the construction. Note that this construction is absolute for $L_{\beta_\gamma+\omega}$.

Clearly $\mathcal{A}$ is an a.d. family, and if $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ with $|F| < |\mathcal{A}| = \aleph_1$, then there is a $\beta < \omega_1$ such that $F \subseteq L_\beta$. Now if F is not finitely covered by $\mathcal{A}$ then for every $f \in F$ and every γ with $\beta_\gamma \geq \beta$ the set $f \cap g_\gamma$ is infinite, which shows that $\mathcal{A}$ is a very mad family.

Now what remains to be seen is that this $\mathcal{A}$ is Π^1_1 definable.

LEMMA 3.6. *If $(\mathbb{N}, E) \cong (L_\alpha, \in)$ and $g \in L_{\alpha+\omega}$ encodes E as in Lemma 3.2, then there is a formula φ only containing quantifiers over the natural numbers such that*

$$\begin{aligned} \varphi(\langle E_\omega, r, u\rangle, g) \Leftrightarrow\ & (\mathbb{N}, E_\omega) \cong (L_{\alpha+\omega}, \in)\ \wedge \\ & r \textit{ is the satisfaction relation for } (\mathbb{N}, E_\omega)\ \wedge \\ & u = \mathsf{Real}_{E_\omega}(g). \end{aligned}$$

PROOF. In the definition below we refer directly to E; that this can be replaced by g is easy.

We define

$$\begin{aligned} \varphi(\langle E_\omega, r, u\rangle, g) \equiv\ & \mathsf{Sat}(E_\omega, r)\ \wedge\ \mathsf{EonEvens}(E_\omega, E)\ \wedge \\ & \mathsf{Levels}(E_\omega, E, r)\ \wedge\ \mathsf{Real}_{E_\omega}(g) = u, \end{aligned}$$

where:

Sat: The formula $\mathsf{Sat}(E_\omega, r)$ states that r is the satisfaction relation for E_ω: (sketch)

$$\begin{aligned} r(\langle \ulcorner\varphi\urcorner, \bar{m}\rangle) = 1 \Leftrightarrow &(\ulcorner\varphi\urcorner = \ulcorner x = y\urcorner \wedge m_0 = m_1) \ \vee \\ &(\ulcorner\varphi\urcorner = \ulcorner x \in y\urcorner \wedge E_\omega(m_0, m_1)) \ \vee \\ &(\ulcorner\varphi\urcorner = \ulcorner\forall x\psi(x)\urcorner \wedge \forall n\ r(\langle\ulcorner\psi\urcorner, \langle n, \bar{m}\rangle\rangle) = 1) \ \vee \\ &(\ulcorner\varphi\urcorner = \ulcorner\neg\psi\urcorner \wedge r(\langle\ulcorner\psi\urcorner, \bar{m}\rangle) = 0) \ \vee \\ &(\ulcorner\varphi\urcorner = \ulcorner\psi_1 \vee \psi_2\urcorner \wedge (r(\langle\ulcorner\psi_1\urcorner, \bar{m}\rangle) = 1 \vee r(\langle\ulcorner\psi_2\urcorner, \bar{m}\rangle) = 1)) \end{aligned}$$

$\mathsf{EonEvens}$: $\mathsf{EonEvens}(E_\omega, E)$ states that E is isomorphic to an initial segment of E_ω and lives on the even natural numbers.

$$\mathsf{EonEvens}(E_\omega, E) \equiv \forall i, j \left(\neg(2i+1\ E_\omega\ 2j) \ \wedge \ (2i\ E_\omega\ 2j \leftrightarrow i\ E\ j)\right)$$

Levels: Here we need a bijection $\pi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that $\pi(0,0) = 1$ and $\pi(0, k+1)$ enumerates the evens; we can easily find such a bijection which is recursive.

Then $\mathsf{Levels}(E_\omega, E, r)$ is the conjunction of $\mathsf{SLevels}(E_\omega, E)$ and $\mathsf{ELevels}(E_\omega, E, r)$ where $\mathsf{SLevels}$ states $\pi(l, 0)$ is the l-th level after $(\mathbb{N}, E)$:

$$\begin{aligned} &\forall l, i, j \left(\left[\, i < l \to \pi(i,j)\ E_\omega\ \pi(l,0) \right] \ \wedge \ \pi(l, j+1)\ E_\omega\ \pi(l, 0) \right) \wedge \\ &\forall l, i, j \left(\pi(i,j)\ E_\omega\ \pi(l,0) \to (i < l \vee (i = l \wedge j \geq 1))\right), \end{aligned}$$

and $\mathsf{ELevels}(E_\omega, E, r)$ that $k \mapsto \pi(l, k+1)$ is an enumeration of the new sets at the l-th level after $(\mathbb{N}, E)$. First we find an enumeration, $k \mapsto \mathsf{ge}(l,k)$, of formulas and parameters that can be used to define sets at the l^{th} level:

Let S be the set $\{(n, \bar{x}) \mid n$ is the Gödel number of a formula with $\mathrm{lh}(\bar{x}) + 1$ free variables $\wedge\ \bar{x} \in {}^{<\mathbb{N}}\mathbb{N}\}$. Then define $\mathsf{ge} : \mathbb{N}\times\mathbb{N} \to S$ such that $\mathsf{ge}[\{(l,k) \mid k \in \mathbb{N}\}] = \{(n, \bar{x}) \in S \mid \bar{x} \in {}^{<\mathbb{N}}(\{\pi(l, k+1) \mid k \in \mathbb{N}\} \cup \{\pi(j,k) \mid j < l \ \wedge\ k \in \mathbb{N}\})\}$. Such a function ge can clearly be chosen to be recursive.

We want to define $\tilde{\mathsf{ge}} : \mathbb{N}\times\mathbb{N} \to S$ such that $k \mapsto \tilde{\mathsf{ge}}(l,k)$ enumerates only the data needed to define new sets at level $l+1$, and does so without repetition. For this we do some preliminary work.

First note that $(\pi(l,0), E_\omega) \models \varphi(x)$ is equivalent to $(\mathbb{N}, E_\omega) \models (\varphi(x))^{\pi(l,0)}$ which in turn is equivalent to $r(\langle\ulcorner(\varphi)^{\pi(l,0)}\urcorner, x\rangle = 1$. The map $\mathsf{rel} : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by $(\ulcorner\varphi\urcorner, l) \mapsto \ulcorner(\varphi)^{\pi(l,0)}\urcorner$ and 0 if the first component of the input is not the Gödel number of a formula is recursive.

Then we define a formula $\mathsf{new}(n, \bar{x}, l)$ such that it is true of $(n, \bar{x}, l)$ iff $n = \ulcorner\varphi\urcorner$ and $\{y\ E_\omega\ \pi(l,0) \mid (\pi(l,0), E_\omega) \models \varphi(\bar{x}, y)\}$ is different from all $\pi(j,k)$ for $j < l$ and $k \in \mathbb{N}$, or $j = l$ and $k > 0$. This means that the set determined by $(n, \bar{x})$ didn't exist at level l (and is not the collection of all sets before level l, which is $\pi(l,0)$). The formula expressing this is:

$$\begin{aligned} \mathsf{new}(n, \bar{x}, l) \equiv \forall j, k\ j \leq l \to \exists j', k' \big\{ &j' \leq l \ \wedge \\ &\left[\left(\pi(j',k')\ E_\omega\ \pi(j,k) \wedge r(\langle \mathsf{rel}(n,l), \langle \bar{x}, \pi(j',k')\rangle\rangle) = 0\right) \ \vee \right. \\ &\left. \left(\neg\pi(j',k')\ E_\omega\ \pi(j,k) \wedge r(\langle \mathsf{rel}(n,l), \langle \bar{x}, \pi(j',k')\rangle\rangle) = 1\right) \right] \big\} \end{aligned}$$

We also need a formula $\mathsf{nb}(l,m)$ that is true of (l,m) iff the set defined by $\mathsf{ge}(l,m)$ from $\pi(l,0)$ is not also defined by $\mathsf{ge}(l,m')$ with $m' < m$.

$$\begin{aligned}\mathsf{nb}(l,m) \equiv \forall m' < m \exists j,k(j < l \vee (j = l \wedge k > 0)) \ \wedge \\ \big[\ \big(r(\langle \mathsf{rel}(\pi_0(\mathsf{ge}(l,m)),l), \langle \pi_1(\mathsf{ge}(l,m)), \pi(j,k)\rangle\rangle) = 1 \ \wedge \\ r(\langle \mathsf{rel}(\pi_0(\mathsf{ge}(l,m')),l), \langle \pi_1(\mathsf{ge}(l,m')), \pi(j,k)\rangle\rangle) = 0 \big) \ \vee \\ \big(r(\langle \mathsf{rel}(\pi_0(\mathsf{ge}(l,m)),l), \langle \pi_1(\mathsf{ge}(l,m)), \pi(j,k)\rangle\rangle) = 0 \ \wedge \\ r(\langle \mathsf{rel}(\pi_0(\mathsf{ge}(l,m')),l), \langle \pi_1(\mathsf{ge}(l,m')), \pi(j,k)\rangle\rangle) = 1 \big) \ \big]\end{aligned}$$

Now we can define $\tilde{\mathsf{ge}}$:

$$\begin{aligned}\tilde{\mathsf{ge}}(l,0) &= \mathsf{ge}(l,k) \text{ for } k \text{ the least number such that} \\ &\qquad (n,\bar{x}) = \mathsf{ge}(l,k) \text{ defines a new set} \\ &= \mathsf{ge}(l,k) \text{ for } k \text{ the least number such that for} \\ &\qquad (n,\bar{x}) = \mathsf{ge}(l,k) \text{ we have } \mathsf{new}(n,\bar{x},l)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathsf{ge}}(l,m+1) &= \mathsf{ge}(l,k) \text{ for } k \text{ the least number such that } (n,\bar{x}) = \mathsf{ge}(l,k) \\ &\quad \text{defines a new set that is not already defined by } \mathsf{ge}(l,\tilde{k}) \\ &\quad \text{with } \tilde{k} \text{ less than or equal to the } k \text{ used in } \tilde{\mathsf{ge}}(l,m) \\ &= \mathsf{ge}(l,k) \text{ for } k \text{ the least number such larger than } \tilde{\mathsf{ge}}(l,m) \\ &\quad \text{such that for } (n,\bar{x}) = \mathsf{ge}(l,k) \text{ we have } \mathsf{new}(n,\bar{x},l) \wedge \mathsf{nb}(l,k)\end{aligned}$$

Now the formula $\mathsf{ELevels}$ can be defined:

$$\begin{aligned}&\forall l,k[\pi(l+1,k+1) \text{ is defined from } \pi(l,0) \\ &\qquad \text{by the formula and parameters in } \tilde{\mathsf{ge}}(l,k)] \\ &\Leftrightarrow \forall l,k,n,\bar{x}((n,\bar{x}) = \tilde{\mathsf{ge}}(l,k) \rightarrow \\ &\qquad [\forall y r(\langle \mathsf{rel}(n,l), \langle \bar{x},y\rangle\rangle) = 1 \leftrightarrow y \ E_\omega \ \pi(l+1,k+1)]\end{aligned}$$

⊣

Note that with these formulas, if $(\mathbb{N},E)$ is wellfounded, then so is $(\mathbb{N},E_\omega)$ (which is the main reason for the lemma to be done the way it is).

Let $\xi_s \in \Sigma_1$ and $\xi_p \in \Pi_1$ be the formulas witnessing that the class $H = \{(x,\gamma) \mid x = L_\gamma\}$ is uniformly $\Delta_1^{L_\alpha}$ for $\alpha > \omega$ a limit ordinal (see [2, Lemma II.2.7]: the proof of this lemma uses some results from earlier in the book which are not correct, but in [8] (Proposition 10.37 on page 213 and its proof) it is shown that there is a theory which is strong enough to prove these results and which is true at L_α for α a limit ordinal).

Let $E \subseteq \mathbb{N} \times \mathbb{N}$ be such that $(\mathbb{N}, E)$ is wellfounded, and let r be its satisfaction relation. Then let $\chi(E, r)$ be the formula (we write $(\mathbb{N}, E) \models \theta$ for $r(\ulcorner\theta\urcorner)$)

$$\begin{gathered}\forall n, m \in \mathbb{N}\ \big[\, (\mathbb{N}, E) \models \text{“}n \text{ is an ordinal”} \ \rightarrow \\ (\mathbb{N}, E) \models \xi_{\mathrm{p}}(m, n) \leftrightarrow (\mathbb{N}, E) \models \xi_{\mathrm{s}}(m, n) \,\big] \wedge \\ (\mathbb{N}, E) \models \ \text{“there is no largest ordinal”} \ \wedge \\ \exists n \in \mathbb{N}\ (\mathbb{N}, E) \models \text{“}n = \omega\text{”} \ \wedge \\ (\mathbb{N}, E) \models \forall x \exists y \left(y \text{ is an ordinal } \wedge \forall z \left(\xi_{\mathrm{p}}(z, y) \rightarrow x \in z\right)\right).\end{gathered}$$

Then the image X of the Mostowski collapse of $(\mathbb{N}, E)$ satisfies that H is Δ^X_1, there is no largest ordinal, $\omega \in X$, and $V = L$. This gives us that $X = (V)^X = (L)^X = L_\alpha$ for $\alpha = X \cap \mathsf{Ord}$ a limit ordinal $> \omega$.

Lemma 3.7.

$$\begin{gathered} g \in \mathcal{A} \Leftrightarrow \textit{the model encoded in } g \textit{ is wellfounded } \wedge \\ \forall \langle E_\omega, r, u\rangle\ \varphi(\langle E_\omega, r, u\rangle, g) \wedge \chi(E_\omega, r) \rightarrow r(\ulcorner u \in \mathcal{A}\urcorner, \bar{\emptyset}) = 1.\end{gathered}$$

Proof. By induction on $\gamma < \omega_1$ we show that for all reals in L_{β_γ} the equivalence holds. So assume that $g \in L_{\beta_\gamma}$ and for all $\gamma' < \gamma$ we have the equivalence for all reals in $L_{\beta_{\gamma'}}$.

If $g \in \mathcal{A}$, then g uniformly encodes $(\mathbb{N}, E)$ such that $(\mathbb{N}, E) \cong (L_{\beta_{\gamma'}}, \in)$ with $\gamma' < \gamma$. The unique model $(\mathbb{N}, E_\omega)$ satisfying $\varphi(\langle E_\omega, r, u\rangle, g)$ has $(\mathbb{N}, E_\omega) \cong (L_{\beta_{\gamma'}+\omega}, \in)$, so also satisfies χ. And in the description of the construction we have shown that $(L_{\beta_{\gamma'}+\omega}, \in) \models g \in \mathcal{A}$, i.e. $(\mathbb{N}, E_\omega) \models \ulcorner u \in \mathcal{A}\urcorner$ where u represents g in the model.

If the model encoded by g is wellfounded and we have

$$\forall \langle E_\omega, r, u\rangle\ \varphi(\langle E_\omega, r, u\rangle, g) \wedge \chi(E_\omega, r) \rightarrow r(\ulcorner u \in \mathcal{A}\urcorner, \bar{\emptyset}) = 1,$$

then the unique $\langle E_\omega, r, u\rangle$ for which $\varphi(\langle E_\omega, r, u\rangle, g)$ has that $(\mathbb{N}, E_\omega)$ is wellfounded and satisfies $\chi(E_\omega, r)$. So there is a countable limit $\beta > \omega$ such that $(\mathbb{N}, E_\omega) \cong (L_\beta, \in)$. Since $(\mathbb{N}, E_\omega) \models u \in \mathcal{A}$, we have $(L_\beta, \in) \models g \in \mathcal{A}$, which by absoluteness gives $g \in \mathcal{A}$. ⊣

Since the formula on the right hand side of the equivalence is clearly Π^1_1, this completes the proof of the theorem.

§4. Questions. In this paper we were concerned with strongly mad families. The results in this paper also answer the corresponding question for very mad families.

Definition 4.1. An eventually different family of functions $\mathcal{F} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *very maximal almost disjoint* (very mad) family iff for every $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ such that $|F| < |\mathcal{F}|$ and no member of F is finitely covered by $\mathcal{F}$, there is a $g \in \mathcal{F}$ such that for all $f \in F$ the set $f \cap g$ is infinite.

In the second section the result is stronger than the corresponding result for very mad families, and in the third section since we were in the context of the continuum hypothesis the notions of very mad and strongly mad agree.

For most types of almost disjoint families a standard axiom of choice construction suffices to construct them in the context of ZFC. This is not true for either strongly or very mad families, which leads to the following question.

QUESTION 4.2. Do strongly and very mad families exist on the basis of ZFC?

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