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THE COMPLEXITY OF MAXIMAL COFINITARY GROUPS

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Abstract. A cofinitary group is a subgroup of the infinite symmetric group in which each element of the subgroup has at most finitely many fixed points. A maximal cofinitary group is a cofinitary group that is maximal with respect to inclusion. We investigate the possible complexities of maximal cofinitary groups, in particular we show that (1) under the axiom of constructibility there exists a coanalytic maximal cofinitary group, and (2) there does not exist an eventually bounded maximal cofinitary group. We also suggest some further directions for investigation.

1. Introduction

In this paper we study maximal cofinitary groups; these are almost disjoint families maximal under the additional requirement that they form a group. An *almost disjoint family* is a family of countable sets (here functions, which are identified with their graph as a subset of $\mathbb{N} \times \mathbb{N}$) such that any two members of the family have finite intersection. A *maximal almost disjoint family (mad family)* is an almost disjoint family not properly contained in another almost disjoint family. Here we do not look at just almost disjoint families, we require the families to form a group as well.

Much work has already been done on the structure and cardinal invariants related to maximal cofinitary groups (see e.g. Adeleke [A81], Truss [T86], Brendle, Spinas, and Zhang [BSZ00], Hrusak, Steprāns, and Zhang [HSZ01], etc.). There is also some previous work on the complexity of these groups for example Gao, Zhang [GZxx]. For a general survey of cofinitary groups see Cameron [C96].

Definition 1.
(i) $\mathrm{Sym}(\mathbb{N})$ is the group of bijections $\mathbb{N} \to \mathbb{N}$.
(ii) $f \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff it has only finitely many fixed points, or is the identity.
(iii) $G \leq \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff all of its members are cofinitary.
(iv) $G \leq \mathrm{Sym}(\mathbb{N})$ is a *maximal cofinitary group* iff it is a cofinitary group and not properly contained in another cofinitary group.

To see that a cofinitary group is an almost disjoint family, let f, g be members of the group, then fixed points n of $g^{-1}f$ correspond to numbers n such that $f(n) = g(n)$. The existence of maximal cofinitary groups follows from Zorn's Lemma: if $\langle G_\alpha \mid \alpha < \beta \rangle$ is an $\subseteq$-increasing chain of cofinitary groups, then $\bigcup_{\alpha<\beta} G_\alpha$ is a

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cofinitary group that is an upper bound for the chain. Zorn's lemma then ensures the existence of a maximal cofinitary group.

Under additional assumptions there are more ways to construct a maximal cofinitary group. Using the following definitions and lemmas one can, for instance, construct a maximal cofinitary group from the continuum hypothesis or Martin's axiom.

Definition 2. (i) If $G \leq \mathrm{Sym}(\mathbb{N})$, then $W_G = G * F(x)$, the free product of G with the free group on one generator.

(ii) If $p, q : \mathbb{N} \rightharpoonup \mathbb{N}$ are finite injective functions, with $p \subseteq q$ we call q a *good extension* of p w.r.t. $w(x) \in W_G$ iff for each $n \in \mathbb{N}$ such that $w(q)(n) = n$, for which $w(p)(n)$ is not defined, there exist $u, z \in W_G$, $l \in \mathbb{N}$ such that
- $w = u^{-1}zu$ with no cancellation,
- $z(p)(l) = l$, and
- $u(q)(n) = l$.

(iii) If $w \in W_G$, then $w = g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_l} g_{l+1}$. The *length of* w, $\mathrm{lh}(w)$, is defined to be $l + 1 + \Sigma_{i=0}^{l} k_i$. If $n \leq \mathrm{lh}(w)$, we define $w \restriction n$ to be the subword of w of length n starting from the right (if $w = gx^2h$, then $w \restriction 1 = h$, $w \restriction 2 = xh$, etc).

(iv) For $p : \mathbb{N} \rightharpoonup \mathbb{N}$ a partial function, $w(x) \in W_G$, and $n \in \mathbb{N}$, we define the *evaluation path* for n in $w(p)$ to be the sequence $\langle l_i \in \mathbb{N} \mid i \leq j \rangle$, with $l_0 = n$, $l_{i+1} = (w \restriction i)(p)(n)$ and

$$j := \begin{cases} \mathrm{lh}(w), & \text{if } w(p)(n) \text{ is defined;} \\ \max\{i \mid (w \restriction i)(p)(n) \text{ is defined}\}, & \text{otherwise} \end{cases}$$

The use of these definitions comes from the following lemmas that can be found in, for instance, Zhang [Z00] and Zhang [Z03].

Lemma 3. *If G is a countable cofinitary group, $p : \mathbb{N} \rightharpoonup \mathbb{N}$ a finite injective function, $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary, and $w \in W_G$ then*

- *(Domain Extension Lemma) for each $n \in \mathbb{N} \setminus \mathrm{dom}(p)$, for all but finitely many $k \in \mathbb{N}$, the extension $p \cup \{(n, k)\}$ is a good extension of p w.r.t. w,*
- *(Range Extension Lemma) for each $k \in \mathbb{N} \setminus \mathrm{ran}(p)$ for all but finitely many $n \in \mathbb{N}$, the extension $p \cup \{(n, k)\}$ is a good extension of p w.r.t. w,*
- *(Hitting f Lemma) for all but finitely many $n \in \mathbb{N}$ the extension $p \cup \{(n, f(n))\}$ is a good extension of p w.r.t. w.*

It is often the case that families of reals where the axiom of choice (Zorn's Lemma is equivalent to this) is used in their existence proof do not have simple descriptions. For example Mathias [M77] showed that there is no analytic mad family of subsets of $\mathbb{N}$ (note that these families can be constructed from the continuum hypothesis or Martin's axiom in much the same way as maximal cofinitary groups can be constructed). Miller [M89] showed that under the axiom of constructibility there exists a coanalytic mad family of subsets of $\mathbb{N}$.

Miller's method is flexible enough to work for a variety of almost disjoint families. The construction of mad families usually proceeds by adding one new member at a time. We recursively construct the family to be $\mathcal{A} = \{f_\alpha \mid \alpha < \omega_1\}$. Then under the axiom of constructibility it is sufficient to prove a coding lemma of the following form.

Lemma 4 (Coding Lemma — Generic Form)**.** *If A is a countable almost disjoint family and $z \in 2^{\mathbb{N}}$, we can construct a new member f to adjoin to the family such that*

(i) *z is recursive in f, and*
(ii) *if we iterate the construction ω_1 many times we construct a maximal almost disjoint family.*

In fact the construction has to be such that z is uniformly recursive in f; the function computing z from f should not depend on A or on other parameters in the construction.

The method in outline is then as follows. There is a collection of levels L_{β_α}, $\alpha < \omega_1$ and $\beta_\alpha < \omega_1$ that are equal to their own Skolem hull under enough of their definable Skolem functions. These levels also have the property that a relation $E \subseteq \mathbb{N} \times \mathbb{N}$ such that $(L_{\beta_\alpha}, \in) \cong (\mathbb{N}, E)$ is in $L_{\beta_\alpha+\omega}$. We can then use the coding lemma to encode E into the next element f_α of the almost disjoint family we construct; this element will also be in $L_{\beta_\alpha+\omega}$. Since from E we can obtain a relation isomorphic to $L_{\beta_\alpha+\omega}$ and $L_{\beta_\alpha+\omega}$ contains all information needed to compute f_α, the function f_α contains "a certificate" for its own membership in the family. Decoding this certificate takes a Π^1_1 formula.

In Miller [M89] there are more details, but in Kastermans [K06] and in Kastermans, Steprāns, and Zhang [KSZxx] full details are given.

Using this method Su Gao and Yi Zhang were able to prove the following in [GZxx].

Theorem 5. *The axiom of constructibility implies that there exists a maximal cofinitary group with a coanalytic generating set.*

They prove a nice version of the generic type coding lemma, and the generating set is constructed in the right way for the general method to apply.

The difficulty in showing that the whole group can be coanalytic is that when you add a new generator you also add countably many other new elements. In the construction we will use the method of good extensions, which implies that the new generator will be free over all that came before. Then for all $w \in W_G \setminus G$ we will have $w(g) \notin G$. And all these elements need to encode E for the method to work (for simplicity we will work with $z \in 2^{\mathbb{N}}$, where z is $n \mapsto \chi_{E(f(n))}$ for $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ a recursive bijection).

The following proposition shows that in the case of cofinitary groups we cannot get z uniformly recursive in f in the coding lemma, when our construction is computable. This does not quite prove that uniform computability is not possible as the construction does not need to be computable, and moreover, it only has to work over a fixed group (the one constructed in previous steps of the construction).

Proposition 6. *There do not exist recursive functionals $\Psi(X, Z, n)$ and $\Phi(X, n)$ such that for all countable cofinitary groups G, and all $z \in 2^{\mathbb{N}}$ the function $g \in \mathrm{Sym}(\mathbb{N})$ defined by $g(n) = \Psi(G, z, n)$ satisfies that $\langle G, g\rangle$ is cofinitary, and for all $w \in W_G$ we have that $z(n) = \Phi(w(g), n)$.*

Proof. Let G be given to us as a countable sequence $\langle g_i \mid i \in \mathbb{N}\rangle$, and assume that Ψ and Φ as in the statement do exist.

Pick a countable cofinitary group G, and a $z \in 2^{\mathbb{N}}$ with $z(0) = 0$. Define g from G and z using Ψ as in the statement of the proposition. Let $u = \mathrm{use}(\Phi, g, 0)$, the use of g by the functional Φ when calculating $\Phi(g,0)$ (that is u is the least number such that all queries Φ makes of g are in $g \restriction u$). Then for all $h \in \mathrm{Sym}(\mathbb{N})$, if $h \restriction u = g \restriction u$ then $\Phi(h,0) = 0$.

Let $z' \in 2^{\mathbb{N}}$ be such that $z'(0) = 1$. Define g' from G and z' using Ψ as in the statement of the lemma. Let $U = \mathrm{use}(\Psi, G, z', g' \restriction u)$, the maximal use of G and z by Ψ in calculating $g'(0), \ldots, g'(u-1)$.

So in determining $g'(0), \ldots, g'(u-1)$ no use is made of any group element in the enumeration $\langle g_i \mid i \in \mathbb{N} \rangle$ with index $i > U$.

Now pick a new cofinitary group $\bar{G}$ and enumeration of it $\langle \bar{g}_i \mid i \in \mathbb{N} \rangle$ such that $\bar{g}_i = g_i$ for $i \leq U$ and there are elements g_l and g_k such that $g_l(g' \restriction u)g_k = g \restriction u$.

Define g'' from $\bar{G}$ and z' using Ψ as in the statement of the proposition. Then $g'' \restriction u = g' \restriction u$. However if we choose $w(x) = g_l x g_k$, then $w(g'') = g \restriction u$, which means that $\Phi(w(g''), 0) = 0$ contradicting the fact that Φ computes z' from $w(g'')$. □

In the next section we show how to overcome this difficulty and prove that the axiom of constructibility implies that there exists a coanalytic maximal cofinitary group.

This does not completely answer the question of what the lowest possible complexity of maximal cofinitary groups is as there is no result yet corresponding to Mathias' result (which shows that there are no analytic maximal almost disjoint families of subsets of $\mathbb{N}$). Blass has observed that if there is an analytic maximal cofinitary group, there is a Borel maximal cofinitary group. From Mathias result the conjecture is that there are no Borel maximal cofinitary groups. The currently known results are far from showing this though.

It is immediate that maximal cofinitary groups cannot be open (any basic open set contains two elements with infinitely many agreements). It is, however, still open if there exists a closed maximal cofinitary group. We show in the last section that maximal cofinitary groups cannot be contained in a K_σ (a countable union of compact sets).

One of the problems in approaching this question is that the methods for constructing maximal cofinitary groups are not very flexible. The method using good extensions always gives rise to free groups; and it is certainly imaginable that the least possible complexity of freely generated maximal cofinitary groups is not the same as the least possible complexity of maximal cofinitary groups in general. Therefore we ask what the possible isomorphism types of maximal cofinitary groups are. Using results from Kastermans [Kxxa], Blass has observed that they cannot be Abelian. We don't know of many other restrictions though. So the question is what the possible isomorphism types of maximal cofinitary groups are.

2. A Coanalytic Maximal Cofinitary Group

In this subsection we will prove the following theorem.

Theorem 7. *The axiom of constructibility implies that there exists a coanalytic maximal cofinitary group.*

We will recursively construct the maximal cofinitary group. To make the coding work out though we have to start with a specific countable group.

Let G_0 be the countable cofinitary group generated by h defined as follows:

$$h(n) = \begin{cases} n-2, & \text{if } n \text{ is even and not zero;} \\ n+2, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n = 0. \end{cases}$$

Then there is a formula only involving natural number quantifiers $\phi_{G_0}(x)$ that defines this group as a subgroup of $\mathrm{Sym}(\mathbb{N})$.

The coding method we use has two cases and a parameter. But with these it will be uniform; there exists a recursive functional $\Phi(X, m, \gamma, n)$ such that if z is encoded in f we have that there exist $m \in \mathbb{N}$ and $\gamma \in \{0, 1\}$ such that for all $n \in \mathbb{N}$ we have $z(n) = \Phi(f, m, \gamma, n)$.

The encoding will be as follows; z is encoded in f with parameter $(m, 0)$ iff

$$(k_n, z(n)) = f^n(m), \text{ for some } k_n \in \mathbb{N};$$

z is encoded in f with parameter $(m, 1)$ iff

$$(k_n, z(n)) = f(hf)^n(m), \text{ for some } k_n \in \mathbb{N}.$$

This encoding will be done in the following way. At some point in the construction, we have already constructed a finite approximation p to the new generator g. We then start encoding into a new word $w \in W_G$. Let $w = g_0 x^{k_0} g_1 \cdots x^{k_l} g_{l+1}$ with $g_i \in G$ $(i \leq l+1)$ and $k_i \in \mathbb{Z} \setminus \{0\}$. Pick m such that $g_{l+1}(m) \notin \mathrm{dom}(p) \cup \mathrm{ran}(p)$, and let $\gamma = 0$ if w does not have a proper conjugate subword, $\gamma = 1$ otherwise. We extend p by taking a good extension (see Definition 2.(ii)) with respect to certain words, extending the evaluation path of $w(p)$ for m. We do this until $a = (w \restriction \mathrm{lh}(w) - 2)(m)$ is defined. Then (assuming $k_0 > 0$, the other case is analogous) we choose a b such that $p \cup \{(a, b)\}$ is a good extension with respect to certain words, such that $w(p \cup \{(a, b)\})(m) \in \{(k, z(0)) \mid k \in \mathbb{N}\}$, and such that we can encode $z(1)$ into the next location.

This last requirement is where the two different types of encoding play a role. If w has no proper conjugate subword, then since G is cofinitary there is only finitely much restriction from the requirement that $g_{l+1}(g_0(b)) \notin \mathrm{dom}(p) \cup \mathrm{ran}(p) \cup \{b\}$. If w does have a proper conjugate subword, then we will always have that $g_{l+1}(g_0(b)) = b$. This is why in that case we "twist" by h. The next location we then want to encode in is $h(g_0(b))$ and, again since G is cofinitary, we will have only finitely much restriction from requiring $g_{l+1}(h(g_0(b))) \notin \mathrm{dom}(p) \cup \mathrm{ran}(p) \cup \{b\}$.

With this we have enough information to state and prove the coding lemma for cofinitary groups.

Lemma 8. *Let G be a countable cofinitary group containing G_0, $F \leq \mathrm{Sym}(\mathbb{N}) \setminus G$ a countable family of permutations such that for all $f \in F$ the group $\langle G, f \rangle$ is cofinitary, and $z \in 2^{\mathbb{N}}$. Then there exists g such that $\langle G, g \rangle$ is cofinitary, for all $f \in F$ the set $f \cap g$ is infinite, and z is recursive in $w(g)$ for all $w \in W_G \setminus G$.*

Proof. Since $W_G \setminus G$ is countable, enumerate it by $\langle w_n \mid n \in \mathbb{N} \rangle$, and enumerate F by $\langle f_n \mid n \in \mathbb{N} \rangle$.

Start by setting $g := \emptyset$, $A := \emptyset$ and $\langle c_n \mid n \in \mathbb{N} \rangle$ with all $c_n := \emptyset$. g will be the permutation we construct, so at any time it will be a finite injective function. A is a set of numbers; it is the set of numbers in domain and range that are being used in coding. We have to avoid this set in all steps other than coding steps. It will always be finite and any number will stay in it for only finitely many stages of the construction. $\langle c_n \mid n \in \mathbb{N}\rangle$ is a sequence of which at any time an initial segment will contain triples that hold information on how far we are in the coding, where the coding currently is being done, and how we are coding.

At step $s \in \mathbb{N}$ in the construction we do the following:

- Extend Domain: Set $a := \min\{\mathbb{N}\setminus(\mathrm{dom}(g)\cup A)\}$. By the Domain Extension Lemma (see Lemma 3) , for all but finitely many b the extension $g\cup\{(a,b)\}$ is a good extension of g with respect to all words w_i, $i \le s$. Choose b to be the least such number such that $b \notin A$ and set $g = g \cup \{(a,b)\}$.
- Extend Range: Set $b := \min\{\mathbb{N} \setminus (\mathrm{ran}(g) \cup A)\}$. By the Range Extension Lemma (see Lemma 3), for all but finitely many a the extension $g\cup\{(a,b)\}$ is a good extension of g with respect to all words w_i, $i \le s$. Choose a to be the least such number such that $b \notin A$ and set $g = g \cup \{(b,a)\}$.

Note: these two sub-steps ensure that g will be a permutation of $\mathbb{N}$; no number stays in A long enough to cause problems.

- Hit f: For each $j \le s$ in turn do the following:
  By the Hitting f Lemma (see Lemma 3) , for all but finitely many a the extension $g \cup \{(a, f_j(a))\}$ is a good extension of g with respect to all words w_i, $i \le s$. Choose a to be the least such number such that $a, f_j(a) \notin A$ and set $g = g \cup \{(a, f_j(a))\}$.

Note: this ensures for all $f \in F$ that $f \cap g$ is infinite.

- Coding: For each $j < s$ in turn do the following:
  c_j is a triple (m, l, γ), where m denotes where the coding is taking place, l denotes the next location of z to encode, and γ determines how to encode.
  Let n be the largest number such that $a := (w_j \upharpoonright n)(g)(m)$ is defined. Then $w_j = w' g_j x^k x^\delta (w_j \upharpoonright n)$, where $w' \in W_G$, $g_j \in G$, and $k \ge 0$ if $\delta = 1$ and $k \le 0$ if $\delta = -1$.
  <u>Case $\delta = 1$:</u>
  By the Domain Extension Lemma, for all but finitely many b the extension $g \cup \{(a,b)\}$ is a good extension of g with respect to all words w_i, $i \le s$.
  <u>Subcase $k > 0$:</u>
  Choose b to be the least number such that $b \notin A \cup \mathrm{dom}(p)$, set $g = g \cup \{(a,b)\}$ and replace a in A by b (so a is no longer a member of A but b now is).
  <u>Subcase $k = 0$:</u>
  <u>SubSubcase $w' = w'' x^{\delta'}$ ($\delta' \in \{-1, 1\}$):</u>
  Choose b to be the least number such that $b \notin A$ and $g_j(b) \notin A\cup\mathrm{dom}(g)\cup \mathrm{ran}(g)$ (in fact depending on δ' we only care about avoiding one of $\mathrm{dom}(g)$ or $\mathrm{ran}(g)$). Set $g = g \cup \{(a,b)\}$ and replace a in A by $g_j(b)$.
  <u>SubSubcase $w' = \emptyset$:</u> (This is where the actual coding happens.)
  Choose b to be the least number such that $b \notin A$, $g_j(b) \notin A$, $g(b) \in \{(c, z(l)) \mid c \in \mathbb{N}\}$ and if $\gamma = 0$ $w_0(g_i(b)) \notin A \cup \mathrm{dom}(g) \cup \mathrm{ran}(g) \cup \{b\}$ or if $\gamma = 1$ then $w_0(h(g_i(b))) \notin A \cup \mathrm{dom}(p) \cup \mathrm{ran}(g) \cup \{b\}$.
  The requirements $b \notin A$, $g_j(b) \notin A$, and $w_0(g_j(b)) \notin A\cup\mathrm{dom}(p)\cup\mathrm{ran}(p)$ or $w_0(h(g_j(b))) \notin A \cup \mathrm{dom}(p) \cup \mathrm{ran}(p)$ exclude finitely many possibilities for b. Since G is cofinitary, $w_0(g_j(b)) \neq b$ or $w_0(h(g_j(b))) \neq b$ also excludes

finitely many possibilities. So we can choose $b \in g_j^{-1}[\{(c, z(l)) \mid c \in \mathbb{N}\}]$ satisfying the last condition on b.

Then set $g = g \cup \{(a, b)\}$, replace a in A by $w_0(g_i(b))$ (if $\gamma = 0$) or $w_0(h(g_i(b)))$ (if $\gamma = 1$) and set $c_j := (g_j(b), l + 1, 0)$ (if $\gamma = 0$) or $c_j := (h(g_j(b)), l + 1, 1)$ (if $\gamma = 1$). (γ is set in Extending Coding below and explained on page 5.)

Case $\delta = -1$: The method and (sub)subcases are analogous to the case $\delta = 1$ but using the range extension lemma.

- Extending Coding: If w_s has a proper conjugate subword, set $\gamma = 1$; otherwise set $\gamma = 0$. Then let a be the least number such that if $w_s = w' g_s$, then $g_s(a) \notin \mathrm{dom}(g) \cup \mathrm{ran}(g) \cup A$. Add $g_s(a)$ to A and set $c_s = (a, 0, \gamma)$. This indicates that at the next stage we will start encoding $z(0)$ into location a for w_s.

Note: with the explanation before the lemma this ensures that the coding happens correctly. □

The construction using this coding lemma is now as follows: Let $\langle L_{\beta_\alpha} \mid \alpha < \omega_1 \rangle$ be an increasing enumeration of the levels L_α ($\alpha < \omega_1$) that are isomorphic to their own Skolem hull under enough of the definable Skolem functions for L. Start with G_0. Then recursively construct G_α for $\alpha < \omega_1$. At limit ordinals take unions. To get $G_{\alpha+1}$ from G_α find the least β_α such that $G_\alpha \subseteq L_{\beta_\alpha}$. Find $E \subseteq \mathbb{N} \times \mathbb{N}$ such that $E \in L_{\beta_\alpha + \omega}$ and $(L_{\beta_\alpha}, \in) \cong (\mathbb{N}, E)$. Let $z \in 2^\omega$ be such that E is recursive in z, and let F be the collection of permutations in L_{β_α} such that for all $f \in F$ the group $\langle G_\alpha, f \rangle$ is cofinitary. Then apply Lemma 8 to construct g_α, and set $G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle$. Finally let $G = \bigcup_{\alpha < \omega_1} G_\alpha$.

G is maximal since all constructible reals are in L_{ω_1}. Suppose there is $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary. Then $f \in L_\alpha$ for some $\alpha < \omega_1$, which means that $f \in L_{\beta_\alpha}$ for cofinally many $\alpha < \omega_1$. This means that as some step ζ in the construction f is a member of the family F, which in turn means that $f \cap g_\zeta$ is infinite, which is a contradiction.

Miller's method gives a Π^1_1 formula $\psi(g, m, \gamma)$ such that $g \in G \setminus G_0$ iff there exist m and γ such that $\psi(g, m, \gamma)$. This means that G can be defined by the formula $\psi_{G_0}(g) \wedge \exists m \in \mathbb{N} \exists \gamma \in \{0, 1\} \psi(g, m, \gamma)$, which is clearly a Π^1_1 formula.

3. There Does Not Exist a K_σ Maximal Cofinitary Group

A set is K_σ if it is a countable union of compact sets; every K_σ set is eventually bounded in the following sense.

Definition 9. (i) We write $\forall^* n\, \varphi(n)$ if for all but finitely many $n \in \mathbb{N}$, $\varphi(n)$.
(ii) For $f, g \in {}^{\mathbb{N}}\mathbb{N}$, f is *eventually bounded* by g, written $f <^* g$, if $\forall^* n\, f(n) < g(n)$ (if there exists $k \in \mathbb{N}$ such that for all $l > k$, $f(l) < g(l)$.
(iii) A set $S \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *eventually bounded* if there exists $f \in {}^{\mathbb{N}}\mathbb{N}$ such that for all $g \in S$, $g <^* f$.

Theorem 10. *If G is a cofinitary group that is eventually bounded, then G is not maximal.*

Proof. Let G be a cofinitary group that is eventually bounded. This means G is contained in a set of the form $\{g \in {}^{\mathbb{N}}\mathbb{N} \mid g <^* f\}$ where we can assume $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function with $f(0) > 0$. We will use this bound f to construct an interval partition, with a distinguished point in each interval, that we in turn use to construct $h \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, h\rangle$ is cofinitary.

Define $I = \big\langle \big([i_n, i_{n+1}), p_n\big) \mid n \in \mathbb{N}\big\rangle$ by $i_0 := 0$, $p_n := f(i_n)$, and $i_{n+1} := f(p_n)$. The main property of this sequence is

$$\forall g \in G\ \forall^* n\ g(p_n) \in [i_n, i_{n+1}),$$

which follows easily from the fact that all elements of G are nearly everywhere strictly bounded by f.

Define h by finite approximations: let $h_0 := \emptyset$. Then at step s define h_{s+1} from h_s as follows:

(1) Let $a := \min(\mathbb{N} \setminus \mathrm{dom}(h_s))$, let n be the least number such that for all $l \geq n$ we have $[i_l, i_{l+1}) \cap \big(\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s) \cup \{a\}\big) = \emptyset$, and set $\bar{h}_{s+1} := h_s \cup \{(a, p_n)\}$.
(2) Let $b := \min(\mathbb{N} \setminus \mathrm{ran}(\bar{h}_{s+1}))$, let m be the least number such that for all $l \geq m$ we have $[i_l, i_{l+1}) \cap \big(\mathrm{dom}(\bar{h}_{s+1}) \cup \mathrm{ran}(\bar{h}_{s+1}) \cup \{b\}\big) = \emptyset$, and set $h_{s+1} := \bar{h}_{s+1} \cup \{(p_m, b)\}$.

Note that the a and b used in this construction satisfy $a \leq b \leq a + 1$. To see this first note that the p_n used go alternately into the domain and range, starting with the range. Then by induction it follows that if $a = b$ in an iteration then the least number in $\mathrm{dom}(h_s) \cup \mathrm{ran}(h_s) \setminus a$ is in the range. If $a + 1 = b$, then that least number is in the domain. The note quickly follows from these facts.

The main properties of h are the following (for $n > 0$).

(1) If $a \in [i_n, i_{n+1}) \setminus \{p_n\}$ and $(a, b) \in h$, then $b > i_{n+1}$.
(2) If $b \in [i_n, i_{n+1}) \setminus \{p_n\}$ and $(a, b) \in h$, then $a > i_{n+1}$.
(3) If $(a, p_n), (p_n, b) \in h$, then at most one of a and b is less than i_n.
(4) If $(a_0, b_0), (a_1, b_1) \in h \cup h^{-1}$ and $a_0 < a_1 < b_0$, then $b_1 < a_0$ or $b_1 > b_0$.

The first three of these follow from the observation that for any pair added to h one of the coordinates is a p_n and this p_n is from a later interval than the other coordinate is in (sometimes both coordinates are equal to p_n for some n, but only one is used as such in the construction). The last one follows from the fact that any added pair has one coordinate strictly bigger than any number mentioned before and the note on the order of a and b above.

Taking the first three properties of h together we get that for any $n > 0$ there is at most one pair in $h \cup h^{-1}$ with one coordinate in $[i_n, i_{n+1})$ and the other smaller than i_n. From this we see that if $l < i_n$ and $h^\epsilon(l) \in [i_n, i_{n+1})$ for $\epsilon \in \{-1, +1\}$ then $h^\epsilon(l) = p_n$. Moreover for $m \in [i_n, i_{n+1}) \setminus \{p_n\}$ both $h(m)$ and $h^{-1}(m)$ are bigger than i_{n+1}; this is also the case for $h^\epsilon(p_n)$, but not for $h^{-\epsilon}(p_n) = l$.

Now we show that $\langle G, h\rangle$ is cofinitary. Let us assume, towards a contradiction, that $w(x) = g_0 x^{k_0} g_1 \cdots g_k x^{k_m} g_{m+1} \in W_G$ is such that $w(h)$ has infinitely many fixed points. We can also assume that $g_{m+1} = \mathrm{Id}$, since this only requires conjugation by g_{m+1} and this does not change the number of fixed points.

We normalize the word w further. For this we work above M, the least number such that for all $n \geq M$ and all $g \in G$ appearing in w we have $g(n) < f(n)$. We want a conjugate w' of w of the form $g_l x^{k_l} g_{l+1} \cdots\ g_{m+1} g_0 x^{k_0} \cdots x^{k_{l-1}}$ such that for infinitely many of its fixed points, n, the image after the first application of h (if $k_{l-1} > 0$) or h^{-1} (if $k_{l-1} < 0$) is bigger than n. Such a conjugate w' exists if for infinitely many fixed points we can find a location in the evaluation path where an application of h increases the number. So suppose that you can't do this;

then for all but finitely many fixed points every application of h leads to a smaller number. In this case we can find an n, a fixed point of $w(h)$ such that no point in its evaluation path $\bar{z}$ is less than M and for all i such that $w_i = h^\epsilon$, $\epsilon \in \{+1, -1\}$, we have $z_{i+1} < z_i$ (remember $z_{i+1} = w_i z_i$).

Now since we start in $w(h)$ by applying h we get $z_1 < z_0 = n$. After this we cannot get back to z_0 as any application of a g appearing in w to a number less than or equal to z_1 will lead to a number strictly less than z_0 (z_0 is in the middle of an interval which does not contain z_1 and z_0 is the f image of the start of the interval it is in). And any application of h to a number strictly less than z_0 will lead to a number strictly less than z_1 (follows from the assumption and 4). This contradiction shows that a conjugate w' as desired exists.

We will study this conjugate w' of w; if it can't have infinitely many fixed points neither can w. There are only finitely many points whose evaluation path in $w'(h)$ involves natural numbers less than M. Leave these out of consideration.

Let $\bar{z}$ be the evaluation path of $w'(h)$ on n, a fixed point for this word where the image after the first application of h is bigger than n. There is a least m such that there is an $a \in \mathbb{N}$ such that $z_{m+1} < i_a \leq z_m$.

If for some l we have $z_{l+1} > z_l$ by an application of h (either h or h^{-1}) we have $z_{l+1} = p_b$ for some $b \in \mathbb{N}$. If we now apply h again (the same of h or h^{-1}) we map to a p_m with $p_m > p_b$. So if we are in w' at some x^l, repeatedly applying h, once we start increasing we will keep on increasing.

If after such applications of h where we increase we apply a $g \in G$ as indicated by w', then g doesn't map the element out of the interval it is in (we are working above M' where no element of g appearing in w' can map further than f or f^{-1}).

Now we know that $z_m = p_k$ for some k, z_{m+1} is obtained from z_m by an application of h, z_m is obtained from z_{m-1} by an application of some $g \in G$ and z_{m-1} is obtained from z_{m-2} by an application of h which was increasing. From the last fact in the last sentence we know $z_{m-1} = p_l$ for some l. Since p_l and p_k are in the same interval, $p_l = p_k$ and we have found a fixed point for this $g \in G$.

So we have found from a fixed point of $w'(h)$ a fixed point for some $g \in G$ appearing in w'. Also, any fixed point of a $g \in G$ appearing in w' can only be used in the evaluation path of finitely many points (and only in the evaluation path of one fixed point if g only appears once in w'). From this we see that if $w'(h)$ has infinitely many fixed points, so does some $g \in G$. This is the contradiction we were looking for. □

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