A Locally Finite Maximal Cofinitary Group

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Outline

- Definitions and Basics.
- Motivation.
- The Result and Idea.

Definition

$\mathrm{Sym}(\mathbb{N})$: the group of bijections $\mathbb{N} \to \mathbb{N}$ with operation composition.

$f \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* (sharp group) iff all $g \in G$ are cofinitary.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *maximal cofinitary group (MCG)* iff G is a cofinitary group and is not properly contained in another cofinitary group.

$g \in \mathrm{Sym}(\mathbb{N})$ is cofinitary iff it is either the identity or has only finitely many fixed points.
$G \leq \mathrm{Sym}(\mathbb{N})$ is cofinitary iff all of its elements are.

- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.
- Any cofinitary group is contained in a maximal cofinitary group.
- (Yi Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is possible that there is an MCG G with $|G| = \kappa$.

$g \in \text{Sym}(\mathbb{N})$ is cofinitary iff it is either the identity or has only finitely many fixed points.
$G \leq \text{Sym}(\mathbb{N})$ is cofinitary iff all of its elements are.

A cofinitary group is an almost disjoint (eventually different) family of permutations that is also a group.

$$g^{-1}f(n) = n \Leftrightarrow f(n) = g(n).$$

For $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ maximal almost disjoint know:

Theorem (Miller)

$\mathcal{A}$ can be coanalytic.

Theorem (Mathias)

$\mathcal{A}$ can not be analytic.

Question

What about mcg?

Theorem (Su Gao, Yi Zhang)

$V = L$ implies there is an mcg with a coanalytic generating set.

Theorem

$V = L$ implies there is a coanalytic mcg.

Question

[Anatoly Vershik] Does there exist a nonrecursive cofinitary group with a recursive set of generators?

Answer: yes.

Question

Does there exist an analytic mcg?

Observation (Blass): if there is an analytic mcg, then it is Borel.

Question

Can there exist a Borel mcg?

Theorem (Otmar Spinas)

There does not exist a locally compact mcg.

Theorem

There does not exist an eventually bounded mcg.

Question

Does there exist a closed mcg?

Get better algebraic understanding of mcg.

Theorem

No mcg has infinitely many orbits.

Theorem

MA implies that for any $n, m \in \mathbb{N}$ there exists an mcg with n finite orbits and m infinite orbits.

Question

What is the orbit structure of the diagonal action on $\mathbb{N}^k$ $(k > 1)$?

Constructing MCG

Use CH or MA.

$$G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle = (G_\alpha * F(x))[x := g_\alpha]$$

$$g_\alpha = \bigcup_{s \in \mathbb{N}} g_{\alpha,s}$$

Study

$$w(g_{\alpha,s}) \rightsquigarrow w(g_{\alpha,s+1})$$

good extension: no unavoidable new fixed points.

$$w = u^{-1} v u$$

"Always" gives *free groups*.

Theorem

MA implies there exists an mcg into which every countable group embeds.

Proof is a "cheat":

$$G = *_{\alpha < \mathfrak{c}} G_a.$$

with G_α of all different isomorphism types.

Question

Are the possible complexities of free mcg the same as the possible complexities of all mcg?

Need more methods of construction.

Test question:

Question

Does there exist a locally finite mcg?

Requires: adding elements having relations with previously added elements.

Answer: under MA yes.

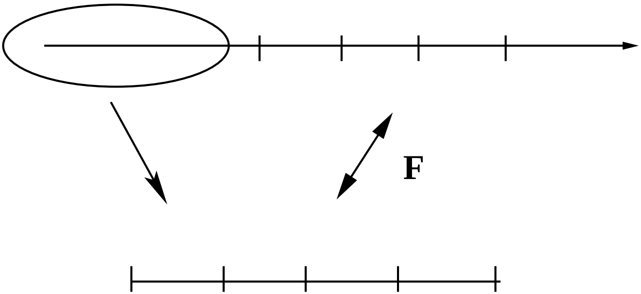
Theorem

MA implies there exists a locally finite mcg.

A finite cg has action on $\mathbb{N}$ with all finite orbits. On all but finitely many of these orbits it acts regularly (transitive and fixed point free).

Any regular action of a group is isomorphic to the group acting on itself.

Have $G = \{g_n : n < \omega\}$ a locally finite cofinitary group.
$f : \mathbb{N} \rightharpoonup \mathbb{N}$ a finite injective map.
Make f act nice with larger and larger parts of G.
Let G_n denote $\langle g_i : i \leq n \rangle$.



Now for the maximality just need the following lemma.

Lemma

Let G be a finite cofinitary group, and $h \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, h \rangle$ is cofinitary. Then for all finite subgroups $H \leq G$ and all but finitely many $n \in \mathbb{N}$ the numbers n and $h(n)$ are not in the same H orbit.

Theorem

MA implies there exists a locally finite mcg.