# Cofinitary Permutation Groups

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April 1, 2005

# Outline

- Definitions and Basics.
- Question of Concrete Example.
- Results.
- Methods.
- Example Theorem.

## Definition

Sym($\mathbb{N}$): the group of bijections $\mathbb{N} \to \mathbb{N}$.

$f \in$ Sym($\mathbb{N}$) is *cofinitary* iff $f$ has only finitely many fixed points.

$G \leq$ Sym($\mathbb{N}$) is a *cofinitary group* iff all $g \in G$ except the identity are cofinitary.

$G \leq$ Sym($\mathbb{N}$) is a *maximal cofinitary group* iff $G$ is a cofinitary group and is not properly contained in another cofinitary group.

### Definition

$f, g \in \mathrm{Sym}(\mathbb{N})$ are almost disjoint iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite.

- A group $G$ is cofinitary iff for all $f, g \in G$ $f$ and $g$ are almost disjoint.
- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.
- Any cofinitary group is contained in a maximal cofinitary group.

Question

(P. Cameron, P. Neumann) If the continuum hypothesis fails, is there an MCG of size $< |\mathbb{R}|$?

(almost nobody believes the continuum hypothesis)

Theorem (Yi Zhang)

*If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is possible that there is an MCG $G$ with $|G| = \kappa$.*

## Question

Does there exist a concrete example of an MCG?

## Question

How definable can an MCG be?

- Borel.
- Topological conditions (closed, compact).
- Projective hierarchy.

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Topology on $\mathrm{Sym}(\mathbb{N})$:
Basic open sets: $\{f \in \mathrm{Sym}(\mathbb{N}) : f \restriction \{0, \ldots, n\} = s\}$ for $s : \{0, \ldots, n\} \to \mathbb{N}$.

### Theorem (Otmar Spinas)

*An MCG can not be locally compact.*

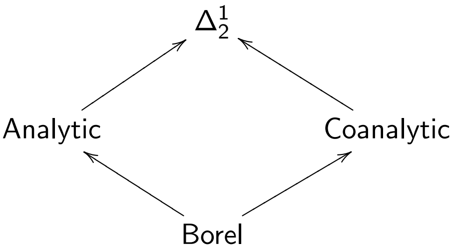
The question if MCG can be closed is still open (tough the conjecture is that they can't be Borel).

$\mathcal{A} \subseteq \mathrm{Sym}(\mathbb{N})$ is locally compact iff for all $f \in \mathcal{A}$ and every neighborhood $O$ of $f$ there is a neighborhood $O'$ of $f$ whose closure $\mathrm{cl}(O')$ is contained in the given neighborhood $O$ such that $\mathrm{cl}(O') \cap \mathcal{A}$ is compact.

Definability:
$\mathcal{A} = \{g \in \mathrm{Sym}(\mathbb{N}) : \varphi(g)\}$ with $\varphi$ a formula.

Complexity of formula gives hierarchy:

- Borel: $\varphi$ only contains quantifiers $\forall n \in \mathbb{N}$ and $\exists n \in \mathbb{N}$ using a real number as a parameter.
- Analytic ($\Sigma^1_1$): $\exists r \in \mathrm{Sym}(\mathbb{N}) \psi(r, g)$ with $\psi$ Borel.
- Coanalytic ($\Pi^1_1$): $\forall r \in \mathrm{Sym}(\mathbb{N}) \psi(r, g)$ with $\psi$ Borel.
- $\Delta^1_2$ is just above analytic and coanalytic: has formula $\forall r_1 \in \mathrm{Sym}(\mathbb{N}) \exists r_2 \in \mathrm{Sym}(\mathbb{N}) \psi(r_1, r_2, g)$ and $\exists r_1 \in \mathrm{Sym}(\mathbb{N}) \forall r_2 \in \mathrm{Sym}(\mathbb{N}) \psi(r_1, r_2, g)$.




Arrows denote inclusions.

Axiom of Constructibility.

Assume the Axiom of Constructability:

By a standard argument you get an MCG of complexity $\Delta^1_2$.

By a complicated combinatorial argument Su Gao and Yi Zhang got the following:

**Theorem (Ax. Constr.)**
*There exists an MCG with a coanalytic generating set.*

One most often grows groups. We get an MCG $G$ by constructing a sequence $G_\alpha \subseteq G_{\alpha+1}$ such that $G = \bigcup_{\alpha<\kappa} G_\alpha$. And then $G_{\alpha+1} = \langle G_\alpha, g\rangle$.

For this need to understand how to add an element $g$.

For any $h \in \langle G, g\rangle$ there is a $w(x) \in G * F(x)$ s.t. $w(g) = h$.

Study $w(g)$ for $w(x) \in G * F(x)$.

Grow the $g$ we want to add: $g = \bigcup_{s\in\mathbb{N}} g_s$, $g_s$ finite injective $\mathbb{N} \rightharpoonup \mathbb{N}$, $g_s \subseteq g_{s+1}$.

Grow the $g$ we want to add: $g = \bigcup_{s\in\mathbb{N}} g_s$, $g_s$ finite injective $\mathbb{N} \rightharpoonup \mathbb{N}$, $g_s \subseteq g_{s+1}$.

$g_{s+1}$ is a good extension of $g_s$ if it does not add fixed points it doesn't have to add.

## Definition

$g_{s+1}$ is a *good extension* of $g_s$ with respect to $w(x) \in G * F(x)$ iff for all $l$ such that $w(g_{s+1})(l) = l$ and $w(g_s)(l)$ is undefined, there exist $u, z, n$ such that

$$w = u^{-1}zu \quad \text{without cancellation}$$
$$z(g_s)(n) = n \quad \text{and} \quad u(g_{s+1})(l) = n$$

Much of the work on MCG is on figuring out how to go from $g_s$ to $g_{s+1}$.

Any group $G \leq \mathrm{Sym}(\mathbb{N})$ has a natural action on $\mathbb{N}$: $(g, n) \mapsto g(n)$.

**Question**

What does this action look like if $G$ is an MCG?

**Theorem**

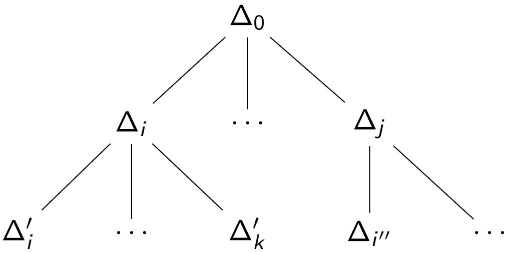
*If $G \leq \mathrm{Sym}(\mathbb{N})$ is cofinitary and has infinitely many orbits, then $G$ is not maximal.*

*Sketch of Proof:*

Let $\Delta_n$, $n \in \mathbb{N}$, be the different orbits.

Let $g \in \mathrm{Sym}(\mathbb{N})$ be such that for all $n$ for all $a \neq b \in \Delta_n$ for all $m, m' \in \mathbb{Z}$ (not both 0) we have that

- $g^m(a) \in \Delta_k$, $k \neq n$,
- $g^{m'}(b) \in \Delta_l$, $l \neq n$ and $l \neq k$.




Where $\Delta_i$ —— $\Delta_j$ means that there is an $a \in \Delta_i$ such that $g(a) \in \Delta_j$ or $g^{-1}(a) \in \Delta_j$. Also we have that for every edge there is a unique pair $(a, g(a))$ or $(a, g^{-1}(a))$ that causes the connection.

Now let $w(x) \in G * F(x)$:

$$w(x) = g_0 x^{l_1} g_1 \cdots x^{l_k} g_k$$
$$w(g) = g_0 g^{l_1} g_1 \cdots g^{l_k} g_k$$

Let $a \in \mathbb{N}$ we'll now see that if $a$ is a fixed point of $w(g)$, we (injectively) find a pair $(g_i, n)$ with $n$ a fixed point for $g_i$. Which will finish the proof as we get a $g_i \in G$ with infinitely many fixed points contradicting that $g_i$ is cofinitary.

$$w(g) = g_0 g^{l_1} g_1 \cdots g^{l_k} g_k$$

$$a, g_k(a) \in \Delta_i \longrightarrow g g_k(a) \in \Delta_j \cdots\!\!\!> g_{m'} g^{k_{m'+1}} \cdots g_k(a) \in \Delta_m$$

$$b \in \Delta_i \rightleftarrows g(b), g_i(g(b)) \in \Delta_j$$

Connection unique: Going from $g(b) \in \Delta_j$ *need* to apply $g^{-1}$. By normal form, first apply a $g_i$, but this $g_i$ cannot move the element as then $g^{-1}$ doesn't bring it back to $\Delta_i$. ⊠

We showed:

**Theorem**

*If $G \leq \mathrm{Sym}(\mathbb{N})$ is cofinitary and has infinitely many orbits, then $G$ is not maximal.*

But the following is still open:

**Question**

Can an MCG have infinitely many orbits under its diagonal action on $\mathbb{N} \times \mathbb{N}$ (or $\mathbb{N}^k$, $k \geq 2$) ?

Peter J. Cameron, *Cofinitary Permutation Groups*, Bull. London Math. Soc. **28** (1996) pp. 113–140.