The Axiom of Constructibility and Π^1_1 Families of Reals

Bart Kastermans

University of Michigan

bart@kastermans.nl
http://www.kastermans.nl/bart/

The usual cumulative hierarchy:

$$V_0 = \emptyset \qquad\qquad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha, \quad \lambda \text{ limit}$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \qquad\qquad V = \bigcup_{\alpha\in\mathsf{Ord}} V_\alpha$$

The constructible hierarchy:

$$L_0 = \emptyset \qquad\qquad L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha, \quad \lambda \text{ limit}$$

$$L_{\alpha+1} = \mathsf{Def}(L_\alpha) \qquad\qquad L = \bigcup_{\alpha\in\mathsf{Ord}} L_\alpha$$

$\mathsf{Def}(L_\alpha)$: all definable sets: for any formula $\phi(x,\bar{z})$ and parameters $\bar{s} \in L_\alpha$

$$\{x \in L_\alpha : L_\alpha \models \phi(x,\bar{s})\} \in \mathsf{Def}(L_\alpha)$$

and not more.

Have: for all $n \leq \omega$: $L_n = V_n$.

$V_{\omega+1}$ contains all reals (or slightly later depending on what a real is to you), whereas all constructible reals are only present at L_{ω_1}.

Def: $f, g \in {}^{\mathbb{N}}\mathbb{N}$ are *almost disjoint* iff $|f \cap g| < \aleph_0$ (iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite).

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *almost disjoint* iff all $f, g \in \mathcal{A}$ are almost disjoint.

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *maximal almost disjoint* (MAD) iff it is almost disjoint and not properly contained in another almost disjoint family.

Thm: (by Arnold Miller) In the constructible universe there exists a MAD family $\mathcal{A}$ that is Π^1_1.

Means: In L

$$\mathcal{A} = \{f \in {}^{\mathbb{N}}\mathbb{N} : \phi(f)\},$$

with $\phi \in \Pi^1_1$ (i.e. $\phi(x) \equiv \forall g \in {}^{\mathbb{N}}\mathbb{N}\ \psi(g, x)$ with ψ having only natural number quantifiers, no function quantifiers).

From now on α is an ordinal less than ω_1.

Lem: If $L_\alpha \cong \mathrm{Sk}(L_\alpha)$ (+technicalities), then there is an $E \subseteq \mathbb{N} \times \mathbb{N}$ such that $E \in L_{\alpha+\omega}$ and $(L_\alpha, \in) \cong (\mathbb{N}, E)$.

Lem: Let $A = \{g_n : n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ be an almost disjoint family, $E \subseteq \mathbb{N} \times \mathbb{N}$ and $F = \{f_n : n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ consist of functions almost disjoint from all functions in A.
Then there exists a function $g : \mathbb{N} \to \mathbb{N}$ almost disjoint from all functions in A, such that E is recursive in g and g hits all f_n infinitely often.
Moreover if $A, E, F \in L_\alpha$, then $g \in L_{\alpha+1}$.

Construction: Inductive.

Let $A = \{g_n : n \in \mathbb{N}\}$ be the countable already constructed almost disjoint family.

Let α be such that $\{g_n : n \in \mathbb{N}\} \in L_\alpha$ and $L_\alpha \cong \mathsf{Sk}(L_\alpha)$.

Let $F \subseteq L_\alpha$ consist of all reals in L_α almost disjoint from A (a countable set).

Let $E \subseteq \mathbb{N} \times \mathbb{N}$ bu such that $(L_\alpha, \in) \cong (\mathbb{N}, E)$.

Then use the lemma to construct a g to add to A.

Now $g \in \mathcal{A}$ iff

1. the model encoded in g is wellfounded.

2. $(\mathbb{N}, E)$ encoded in g is isomorphic to some $(L_\alpha, \in)$ and in $(L_{\alpha+\omega}, \in)$ we can see that $g \in \mathcal{A}$.

Need to get our hands on $L_{\alpha+\omega}$. This we can do: there is a Δ^1_0 formula φ such that

$$\varphi(\overbrace{\langle E_\omega, r, u\rangle}^{\text{all coded in a real}}, g) \Leftrightarrow$$

$$(\mathbb{N}, E_\omega) \cong (L_{\alpha+\omega}, \in) \wedge$$

r is the satisfaction relation of $(\mathbb{N}, E_\omega) \wedge$

u is the element of $(\mathbb{N}, E_\omega)$ playing the role of g

We get:

$$
\begin{aligned}
& g \in \mathcal{A} \\
& \qquad \Leftrightarrow \\
& \qquad\qquad \text{the model encoded in } g \text{ is wellfounded} \wedge \\
& \qquad\qquad \forall \langle E_\omega, r, u \rangle \; \Big[\; \varphi(\langle E_\omega, r, u \rangle, g) \wedge \chi(E_\omega, r) \\
& \qquad\qquad\qquad\qquad\qquad \rightarrow r(\ulcorner u \in \mathcal{A} \urcorner) = 1 \Big]
\end{aligned}
$$

Here χ takes care of very important details. It ensures E_ω is in fact isomorphic to a level of L.

This shows $\mathcal{A}$ is Π^1_1 in the constructible universe as was desired.

Q: Can we get these types of results under weaker hypothesis?

References:

Arnold W. Miller, *Infinite Combinatorics and Definability*, APAL 41 (1989) pp. 179–203.

Bart Kastemans, Yi Zhang, note on very mad families (in preparation).