

The Axiom of Constructibility
and
 Π_1^1 Families of Reals

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The usual cumulative hierarchy:

$$\begin{array}{ll}
 V_0 = \emptyset & V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha, \quad \lambda \text{ limit} \\
 V_{\alpha+1} = \mathcal{P}(V_\alpha) & V = \bigcup_{\alpha \in \text{Ord}} V_\alpha
 \end{array}$$

The constructible hierarchy:

$$\begin{array}{ll}
 L_0 = \emptyset & L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha, \quad \lambda \text{ limit} \\
 L_{\alpha+1} = \text{Def}(L_\alpha) & L = \bigcup_{\alpha \in \text{Ord}} L_\alpha
 \end{array}$$

$\text{Def}(L_\alpha)$: all definable sets: for any formula $\phi(x, \bar{z})$ and parameters $\bar{s} \in L_\alpha$

$$\{x \in L_\alpha : L_\alpha \models \phi(x, \bar{s})\} \in \text{Def}(L_\alpha)$$

and not more.

Have: for all $n \leq \omega$: $L_n = V_n$.

$V_{\omega+1}$ contains all reals (or slightly later depending on what a real is to you), whereas all constructible reals are only present at L_{ω_1} .

Def: $f, g \in {}^{\mathbb{N}}\mathbb{N}$ are *almost disjoint* iff $|f \cap g| < \aleph_0$ (iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite).

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *almost disjoint* iff all $f, g \in \mathcal{A}$ are almost disjoint.

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *maximal almost disjoint* (MAD) iff it is almost disjoint and not properly contained in another almost disjoint family.

Thm: (by Arnold Miller) In the constructible universe there exists a MAD family \mathcal{A} that is Π_1^1 .

Means: In L

$$\mathcal{A} = \{f \in {}^{\mathbb{N}}\mathbb{N} : \phi(f)\},$$

with $\phi \in \Pi_1^1$ (i.e. $\phi(x) \equiv \forall g \in {}^{\mathbb{N}}\mathbb{N} \psi(g, x)$ with ψ having only natural number quantifiers, no function quantifiers).

From now on α is an ordinal less than ω_1 .

Lem: If $L_\alpha \cong \text{Sk}(L_\alpha)$ (+technicalities), then there is an $E \subseteq \mathbb{N} \times \mathbb{N}$ such that $E \in L_{\alpha+\omega}$ and $(L_\alpha, \in) \cong (\mathbb{N}, E)$.

Lem: Let $A = \{g_n : n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ be an almost disjoint family, $E \subseteq \mathbb{N} \times \mathbb{N}$ and $F = \{f_n : n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ consist of functions almost disjoint from all functions in A .

Then there exists a function $g : \mathbb{N} \rightarrow \mathbb{N}$ almost disjoint from all functions in A , such that E is recursive in g and g hits all f_n infinitely often. Moreover if $A, E, F \in L_\alpha$, then $g \in L_{\alpha+1}$.

Construction: Inductive.

Let $A = \{g_n : n \in \mathbb{N}\}$ be the countable already constructed almost disjoint family.

Let α be such that $\{g_n : n \in \mathbb{N}\} \in L_\alpha$ and $L_\alpha \cong \text{Sk}(L_\alpha)$.

Let $F \subseteq L_\alpha$ consist of all reals in L_α almost disjoint from A (a countable set).

Let $E \subseteq \mathbb{N} \times \mathbb{N}$ be such that $(L_\alpha, \in) \cong (\mathbb{N}, E)$.

Then use the lemma to construct a g to add to A .

Now $g \in \mathcal{A}$ iff

1. the model encoded in g is wellfounded.
2. (\mathbb{N}, E) encoded in g is isomorphic to some (L_α, \in) and in $(L_{\alpha+\omega}, \in)$ we can see that $g \in \mathcal{A}$.

Need to get our hands on $L_{\alpha+\omega}$. This we can do: there is a Δ_0^1 formula φ such that

all coded in a real

$$\varphi(\overbrace{\langle E_\omega, r, u \rangle}, g) \Leftrightarrow$$

$$(\mathbb{N}, E_\omega) \cong (L_{\alpha+\omega}, \in) \wedge$$

r is the satisfaction relation of $(\mathbb{N}, E_\omega) \wedge$

u is the element of (\mathbb{N}, E_ω) playing the role of g

We get:

$$g \in \mathcal{A}$$

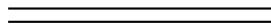
\Leftrightarrow

the model encoded in g is wellfounded \wedge
 $\forall \langle E_\omega, r, u \rangle \left[\varphi(\langle E_\omega, r, u \rangle, g) \wedge \chi(E_\omega, r) \right.$
 $\left. \rightarrow r(\ulcorner u \in \mathcal{A} \urcorner) = 1 \right]$

Here χ takes care of very important details. It ensures E_ω is in fact isomorphic to a level of L .

This shows \mathcal{A} is Π_1^1 in the constructible universe as was desired.

Q: Can we get these types of results under weaker hypothesis?



References:

Arnold W. Miller, *Infinite Combinatorics and Definability*, APAL 41 (1989) pp. 179–203.

Bart Kastemans, Yi Zhang, note on very mad families (in preparation).