

Orthogonal Very Mad Families

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Outline

- ▶ Introduction.
 - ▶ Definition Very Mad Families.
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 - ▶ Definition Orthogonality.
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 - ▶ (CH) To any an orthogonal exists.
- ▶ Questions.
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Mad Families of Functions

Definition

${}^{\mathbb{N}}\mathbb{N}$ is Baire space, the space of functions from \mathbb{N} to \mathbb{N} .

$f, g \in {}^{\mathbb{N}}\mathbb{N}$ are *almost disjoint* (eventually different) iff $f \cap g$ is finite ($\exists k \forall l > k f(l) \neq g(l)$).

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is an *almost disjoint family* iff for all $f, g \in \mathcal{A}$ f and g are almost disjoint.

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *maximal almost disjoint family of functions* (*mad family*) iff \mathcal{A} is an almost disjoint family and is not properly contained in another almost disjoint family.

Existence is immediate from a Zorn's Lemma argument.

Definition

$f \in {}^{\mathbb{N}}\mathbb{N}$ is *finitely covered* by $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff there exist $g_0, \dots, g_n \in \mathcal{A}$ such that $f \setminus \bigcup_{i \leq n} g_i$ is finite.

$F \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *finitely covered* by $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff there exists $f \in F$ such that f is finitely covered by \mathcal{A} .

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *very mad family* iff

- ▶ \mathcal{A} is an almost disjoint family;
- ▶ for every $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ such that $|F| < |\mathcal{A}|$ that is not finitely covered by \mathcal{A} there exists $g \in \mathcal{A}$ such that for all $f \in F$ $f \cap g$ is infinite.

Question

Does there exist an analytic maximal almost disjoint family of functions?

Any very mad family is a mad family of functions.

Theorem (Juris Steprāns)

There does not exist an analytic very mad family.

Theorem (BK and Yi Zhang)

The axiom of constructibility implies that there exists a coanalytic very mad family.

Theorem (BK)

Martin's axiom implies that very mad families exist and are of cardinality continuum.

Theorem (BK)

Any model of ZFC + CH contains a very mad family that is still very mad in any Cohen extension of the model.

Theorem (BK)

Suppose that κ is a regular uncountable cardinal less than the continuum in a model of ZFC. Then there exists a forcing extension preserving cardinals and the cardinality of the continuum, in which there is a very mad family of cardinality κ .

Poset Used

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$: Define $\mathbb{P}_{\mathcal{A}} = \langle P, \leq \rangle$ by

$\langle s, A \rangle \in P$ iff

- ▶ $s : \mathbb{N} \rightarrow \mathbb{N}$ is a finite partial function;
- ▶ $A \subseteq \mathcal{A}$ is finite.

$\langle s_1, A_1 \rangle \leq \langle s_0, A_0 \rangle$ iff

- ▶ $s_0 \subseteq s_1$;
- ▶ $A_0 \subseteq A_1$;
- ▶ for all $f \in A_0$, $f \cap s_1 \subseteq s_0$.

Lemma (MA)

Assume that \mathcal{A} is an almost disjoint family of functions with $|\mathcal{A}| < 2^{\aleph_0}$ and that F is a family not finitely covered by \mathcal{A} with $|F| < 2^{\aleph_0}$. Then there exists a function $g \notin \mathcal{A}$ such that:

- ▶ $\mathcal{A} \cup \{g\}$ is an almost disjoint family of functions;
- ▶ for all $f \in F$, the set $f \cap g$ is infinite.

Dense sets $f \in F$, $h \in \mathcal{A}$ and $n \in \mathbb{N}$:

- ▶ $C_h := \{\langle s, A \rangle \in \mathbb{P}_{\mathcal{A}} : h \in A\}$;
- ▶ $D_n := \{\langle s, A \rangle \in \mathbb{P}_{\mathcal{A}} : n \in \text{dom}(s)\}$;
- ▶ $E_{f,n} := \{\langle s, A \rangle \in \mathbb{P}_{\mathcal{A}} : \exists m \geq n f(m) = s(m)\}$.

Definition

$\mathcal{A}, \mathcal{B} \subseteq {}^{\mathbb{N}}\mathbb{N}$ are *orthogonal* iff \mathcal{A} is not finitely covered by \mathcal{B} and \mathcal{B} is not finitely covered by \mathcal{A} .

Theorem (BK)

Martin's axiom implies there exists \mathcal{A}_α ($\alpha < 2^{\aleph_0}$) all very mad families such that for all $\alpha_1 \neq \alpha_2 < 2^{\aleph_0}$ the families \mathcal{A}_{α_1} and \mathcal{A}_{α_2} are orthogonal.

Notion of forcing: Let $\langle \mathcal{A}_\alpha : \alpha < \beta \rangle$ be subsets of Baire space.

Define $\mathbb{Q}_{\langle \mathcal{A}_\alpha : \alpha < \beta \rangle} = \langle \mathbb{Q}, \leq \rangle$ by

$\bar{p} \in \mathbb{Q}$ iff p is a finite partial function with domain contained in β and $\bar{p}(\alpha) \in \mathbb{P}_{\mathcal{A}_\alpha}$.

$\bar{q} \leq \bar{p}$ iff $\text{dom}(\bar{p}) \subseteq \text{dom}(\bar{q})$ and $\forall \alpha \in \text{dom}(\bar{p}) \bar{q}(\alpha) \leq_{\mathbb{P}_{\mathcal{A}_\alpha}} \bar{p}(\alpha)$.

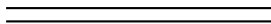
Dense sets: as before plus

$$A_{\alpha_1, \alpha_2, A, n} := \{ \bar{p} \in \mathbb{Q} : A \subseteq \pi_1(\bar{p}(\alpha_2)) \wedge \\ \exists m > n [m \in \text{dom}(\pi_0(\bar{p}(\alpha_1))) \wedge \\ m \in \text{dom}(\pi_0(\bar{p}(\alpha_2))) \wedge \\ \pi_0(\bar{p}(\alpha_1))(m) \notin \{ f(m) : f \in A \} \cup \{ \pi_0(\bar{p}(\alpha_2))(m) \}] \}$$

Theorem (BK)

The continuum hypothesis implies that for every very mad family \mathcal{A} there exists a very mad family \mathcal{B} such that \mathcal{A} and \mathcal{B} are orthogonal.

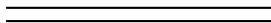
Have \mathcal{A} , want \mathcal{B} orthogonal to it.



In constructing \mathcal{B} need two things:

- ▶ A way to ensure \mathcal{A} does not get finitely covered;
- ▶ A way to ensure \mathcal{B} avoids getting finitely covered.

Have $\text{vmad } \mathcal{A}$, want $\text{vmad } \mathcal{B}$ orthogonal to it.

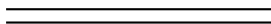


Way to ensure \mathcal{B} avoids getting finitely covered:

Lemma

If $g \in {}^{\mathbb{N}}\mathbb{N}$ is such that there exists $\langle g_n : n \in \mathbb{N} \rangle$ with $\|g_n \in \mathcal{A}$ different and g agrees with each g_n on infinitely many inputs, then g is not finitely covered by \mathcal{A} .

Have $\text{vmad } \mathcal{A}$, want $\text{vmad } \mathcal{B}$ orthogonal to it.
Know how to get \mathcal{B} not finitely covered.



Lemma

Let $\bar{g}_0, \dots, \bar{g}_n \in {}^{\mathbb{N}}\mathbb{N}$, $g_0, \dots, g_n \in \mathcal{A}$ and $W \in [\mathbb{N}]^{\aleph_0}$ such that for all $i \leq n$, $\text{dom}(\bar{g}_i \cap g_i) \supseteq W$. Then

$$g \in \mathcal{A} \wedge g \upharpoonright W \subseteq^* \bigcup_{i \leq n} \bar{g}_i \Rightarrow \text{there is } i \leq n \text{ such that } g = g_i.$$

Corollary

Under the same hypothesis, if $g \in \mathcal{A}$ is not one of the the functions g_0, \dots, g_n then $g \upharpoonright W \setminus \bigcup_{i \leq n} \bar{g}_i$ is infinite.

For a family $\mathcal{B} \subseteq {}^{\mathbb{N}}\mathbb{N}$, a function

$$H : [\mathcal{B}]^{<\aleph_0} \rightarrow ([\mathbb{N}]^{\aleph_0})^2 \times ([\mathcal{A}]^{<\aleph_0})^2$$

is *good for* \mathcal{B} if for all $\{\bar{g}_0, \dots, \bar{g}_n\} \in [\mathcal{B}]^{<\aleph_0}$ we have that $H(\{\bar{g}_0, \dots, \bar{g}_n\}) = \langle W_0, W_1, \{g_i^0, \dots, g_n^0\}, \{g_i^1, \dots, g_n^1\} \rangle$ such that for all $i, j \leq n$ and $k, l \in \{0, 1\}$,

$$i \neq j \vee k \neq l \Rightarrow g_i^k \neq g_j^l,$$

and for all $i \leq n$,

$$\text{dom}(\bar{g}_i \cap g_i^0) \supseteq W_0 \wedge \text{dom}(\bar{g}_i \cap g_i^1) \supseteq W_1.$$

Lemma

If \mathcal{B} is such that an H good for \mathcal{B} exists, then \mathcal{B} does not finitely cover \mathcal{A} .

Question

Do very mad families exist?

Question

Is it consistent that there exists a very mad family with no very mad family orthogonal to it?

Question

Is it consistent that there exist a very mad family of singular size (cofinality ω)?

Question

Orthogonality for other mad type families.

References

Bart Kastermans, *Very Mad Families*, accepted in proceedings of the North Texas Logic Conference, to appear as a special volume of Contemporary Mathematics.

Bart Kastermans, *Cofinitary Groups and Other Almost Disjoint Families*, thesis 2006.

Bart Kastermans, Juris Steprāns, Yi Zhang, *Analytic and Coanalytic Families of Almost Disjoint Functions*.