Orthogonal Very Mad Families

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Outline

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  - Definition Orthogonality.
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Mad Families of Functions

Definition

${}^{\mathbb{N}}\mathbb{N}$ is Baire space, the space of functions from $\mathbb{N}$ to $\mathbb{N}$.

$f, g \in {}^{\mathbb{N}}\mathbb{N}$ are *almost disjoint (eventually different)* iff $f \cap g$ is finite ($\exists k\ \forall l > k\ f(l) \neq g(l)$).

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is an *almost disjoint family* iff for all $f, g \in \mathcal{A}$ f and g are almost disjoint.

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *maximal almost disjoint family of functions (mad family)* iff $\mathcal{A}$ is an almost disjoint family and is not properly contained in another almost disjoint family.

Existence is immediate from a Zorn's Lemma argument.

Definition

$f \in {}^{\mathbb{N}}\mathbb{N}$ is *finitely covered* by $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff there exist $g_0, \ldots, g_n \in \mathcal{A}$ such that $f \setminus \bigcup_{i \leq n} g_i$ is finite.

$F \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *finitely covered* by $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff there exists $f \in F$ such that f is finitely covered by $\mathcal{A}$.

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is a *very mad family* iff

- $\mathcal{A}$ is an almost disjoint family;
- for every $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ such that $|F| < |\mathcal{A}|$ that is not finitely covered by $\mathcal{A}$ there exists $g \in \mathcal{A}$ such that for all $f \in F$ $f \cap g$ is infinite.

Question

Does there exist an analytic maximal almost disjont family of functions?

Any very mad family is a mad family of functions.

Theorem (Juris Steprāns)

There does not exist an analytic very mad family.

Theorem (BK and Yi Zhang)

The axiom of constructibility implies that there exists a coanalytic very mad family.

Theorem (BK)

Martin's axiom implies that very mad families exist and are of cardinality continuum.

Theorem (BK)

Any model of ZFC + CH *contains a very mad family that is still very mad in any Cohen extension of the model.*

Theorem (BK)

Suppose that κ is a regular uncountable cardinal less than the continuum in a model of ZFC*. Then there exists a forcing extension preserving cardinals and the cardinality of the continuum, in which there is a very mad family of cardinality κ.*

Poset Used

$\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$: Define $\mathbb{P}_{\mathcal{A}} = \langle P, \leq \rangle$ by

$\langle s, A \rangle \in P$ iff

- $s : \mathbb{N} \rightharpoonup \mathbb{N}$ is a finite partial function;
- $A \subseteq \mathcal{A}$ is finite.

$\langle s_1, A_1 \rangle \leq \langle s_0, A_0 \rangle$ iff

- $s_0 \subseteq s_1$;
- $A_0 \subseteq A_1$;
- for all $f \in A_0$, $f \cap s_1 \subseteq s_0$.

Lemma (MA)

Assume that $\mathcal{A}$ is an almost disjoint family of functions with $|\mathcal{A}| < 2^{\aleph_0}$ and that F is a family not finitely covered by $\mathcal{A}$ with $|F| < 2^{\aleph_0}$. Then there exists a function $g \notin \mathcal{A}$ such that:

- *$\mathcal{A} \cup \{g\}$ is an almost disjoint family of functions;*
- *for all $f \in F$, the set $f \cap g$ is infinite.*

Dense sets $f \in F$, $h \in \mathcal{A}$ and $n \in \mathbb{N}$:

- $C_h := \{\langle s, A\rangle \in \mathbb{P}_{\mathcal{A}} : h \in A\}$;
- $D_n := \{\langle s, A\rangle \in \mathbb{P}_{\mathcal{A}} : n \in \mathrm{dom}(s)\}$;
- $E_{f,n} := \{\langle s, A\rangle \in \mathbb{P}_{\mathcal{A}} : \exists m \geq n\ f(m) = s(m)\}$.

Definition

$\mathcal{A}, \mathcal{B} \subseteq {}^{\mathbb{N}}\mathbb{N}$ are *orthogonal* iff $\mathcal{A}$ is not finitely covered by $\mathcal{B}$ and $\mathcal{B}$ is not finitely covered by $\mathcal{A}$.

Theorem (BK)

Martin's axiom implies there exists $\mathcal{A}_\alpha$ $(\alpha < 2^{\aleph_0})$ all very mad families such that for all $\alpha_1 \neq \alpha_2 < 2^{\aleph_0}$ the families $\mathcal{A}_{\alpha_1}$ and $\mathcal{A}_{\alpha_2}$ are orthogonal.

Notion of forcing: Let $\langle \mathcal{A}_\alpha : \alpha < \beta \rangle$ be subsets of Baire space. Define $\mathbb{Q}_{\langle \mathcal{A}_\alpha : \alpha < \beta \rangle} = \langle Q, \leq \rangle$ by

$\bar{p} \in Q$ iff p is a finite partial function with domain contained in β and $\bar{p}(\alpha) \in \mathbb{P}_{\mathcal{A}_\alpha}$.

$\bar{q} \leq \bar{p}$ iff $\mathrm{dom}(\bar{p}) \subseteq \mathrm{dom}(\bar{q})$ and $\forall \alpha \in \mathrm{dom}(\bar{p})$ $\bar{q}(\alpha) \leq_{\mathbb{P}_{\mathcal{A}_\alpha}} \bar{p}(\alpha)$.

Dense sets: as before plus

$$\begin{aligned} A_{\alpha_1, \alpha_2, A, n} := \{\bar{p} \in Q : A \subseteq \pi_1(\bar{p}(\alpha_2)) \wedge \\ \exists m > n \, [m \in \mathrm{dom}(\pi_0(\bar{p}(\alpha_1))) \wedge \\ m \in \mathrm{dom}(\pi_0(\bar{p}(\alpha_2))) \wedge \\ \pi_0(\bar{p}(\alpha_1))(m) \notin \{f(m) : f \in A\} \cup \{\pi_0(\bar{p}(\alpha_2))(m)\}]\} \end{aligned}$$

Theorem (BK)

The continuum hypothesis implies that for every very mad family $\mathcal{A}$ there exists a very mad family $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ are orthogonal.

Have vmad $\mathcal{A}$, want vmad $\mathcal{B}$ orthogonal to it.

In contructing $\mathcal{B}$ need two things:

- A way to ensure $\mathcal{A}$ does not get finitely covered;
- A way to ensure $\mathcal{B}$ avoids getting finitely covered.

Have vmad $\mathcal{A}$, want vmad $\mathcal{B}$ orthogonal to it.

Way to ensure $\mathcal{B}$ avoids getting finitely covered:

Lemma

If $g \in {}^{\mathbb{N}}\mathbb{N}$ is such that there exists $\langle g_n : n \in \mathbb{N} \rangle$ with ll $g_n \in \mathcal{A}$ different and g agrees with each g_n on infinitely many inputs, then g is not finitely covered by $\mathcal{A}$.

Have vmad $\mathcal{A}$, want vmad $\mathcal{B}$ orthogonal to it.
Know how to get $\mathcal{B}$ not finitely covered.

Lemma

Let $\bar{g}_0, \ldots, \bar{g}_n \in {}^{\mathbb{N}}\mathbb{N}$, $g_0, \ldots, g_n \in \mathcal{A}$ and $W \in [\mathbb{N}]^{\aleph_0}$ such that for all $i \leq n$, $\mathrm{dom}(\bar{g}_i \cap g_i) \supseteq W$. Then

$$g \in \mathcal{A} \ \wedge g \restriction W \subseteq^* \bigcup_{i \leq n} \bar{g}_i \Rightarrow \text{ there is } i \leq n \text{ such that } g = g_i.$$

Corollary

Under the same hypothesis, if $g \in \mathcal{A}$ is not one of the the functions $g_0, \ldots, g_n$ then $g \restriction W \setminus \bigcup_{i \leq n} \bar{g}_i$ is infinite.

For a family $\mathcal{B} \subseteq {}^{\mathbb{N}}\mathbb{N}$, a function

$$H : [\mathcal{B}]^{<\aleph_0} \to ([\mathbb{N}]^{\aleph_0})^2 \times ([\mathcal{A}]^{<\aleph_0})^2$$

is *good for* $\mathcal{B}$ if for all $\{\bar{g}_0, \ldots, \bar{g}_n\} \in [\mathcal{B}]^{<\aleph_0}$ we have that $H(\{\bar{g}_0, \ldots, \bar{g}_n\}) = \langle W_0, W_1, \{g_0^0, \ldots, g_n^0\}, \{g_0^1, \ldots, g_n^1\}\rangle$ such that for all $i, j \leq n$ and $k, l \in \{0, 1\}$,

$$i \neq j \;\vee\; k \neq l \Rightarrow g_i^l \neq g_j^k,$$

and for all $i \leq n$,

$$\mathrm{dom}(\bar{g}_i \cap g_i^0) \supseteq W_0 \;\wedge\; \mathrm{dom}(\bar{g}_i \cap g_i^1) \supseteq W_1.$$

Lemma

If $\mathcal{B}$ is such that an H good for $\mathcal{B}$ exists, then $\mathcal{B}$ does not finitely cover $\mathcal{A}$.

Question
Do very mad families exist?

Question
Is it consistent that there exists a very mad family with no very mad family orthogonal to it?

Question
Is it consistent that there exist a very mad family of singular size (cofinality ω)?

Question
Orthogonality for other mad type families.

References

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Bart Kastermans, Juris Steprāns, Yi Zhang, *Analytic and Coanalytic Families of Almost Disjoint Functions*.