Cardinal Invariants Related to Permutation Groups

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Contents of talk

- Introduction to basic relavant notions.
- New result on cardinal invariants + ideas from the proof.
- Questions.

$\mathfrak{a}_p := \min\{|F| : F \subseteq \mathrm{Sym}(\mathbb{N})$ is a maximal eventually different family$\}$.

<u>Where:</u>

- $\mathrm{Sym}(\mathbb{N})$ is the group of all permutations of the natural numbers.

- $F \subseteq \mathrm{Sym}(\mathbb{N})$ is an *eventually different* (*almost disjoint*) family iff for all $h_0, h_1 \in F$ the set $\{n \in \mathbb{N} : h_0(n) = h_1(n)\}$ is finite ($\Leftrightarrow h_0 \cap h_1$ finite).

- *maximal*: w.r.t. inclusion, i.e. not properly contained in another eventually different family.

$\mathfrak{a}_g := \min\{|G| : G \leq \mathrm{Sym}(\mathbb{N})$ is a maximal cofinitary group $\}$.

<u>Where:</u>

- $\mathrm{Sym}(\mathbb{N})$ is the group of all permutations of the natural numbers.
- $g \in \mathrm{Sym}(\mathbb{N})$ is cofinitary iff it is either the identity of has only finitely many fixed points.
- $G \leq \mathrm{Sym}(\mathbb{N})$ is cofinitary iff all its members are cofinitary.
- maximal: w.r.t. inclusion, i.e. not properly contained in another cofinitary group.

$\mathfrak{a}_p := \min\{|F| : F \subseteq \mathrm{Sym}(\mathbb{N})$ is a maximal eventually different family$\}$.

$\mathfrak{a}_g := \min\{|G| : G \leq \mathrm{Sym}(\mathbb{N})$ is a maximal cofinitary group $\}$.

<u>Questions:</u>

- How do $\mathfrak{a}_g$ and $\mathfrak{a}_p$ relate to other $\mathfrak{a}$'s and to eachother?

- How do $\mathfrak{a}_g$ and $\mathfrak{a}_p$ relate to other cardinal invariants related to $\mathrm{Sym}(\mathbb{N})$?

$c(\mathrm{Sym}(\mathbb{N}))$ is the least λ such that $\mathrm{Sym}(\mathbb{N})$ can be written as the union of a chain of λ proper subgroups.

- Many interesting questions about cofinitary groups themselves.

Known:

$\mathfrak{a}_g$ is uncountable and can be forced to be anything $\leq \mathfrak{c}$.

$c(\mathsf{Sym}(\mathbb{N}))$ is uncountable and $Con(c(\mathsf{Sym}(\mathbb{N})) < \mathfrak{a}_g)$.

Q: Does ZFC prove $c(\mathsf{Sym}(\mathbb{N})) \leq \mathfrak{a}_g$?

We'll show today that:

A: NO, $Con(c(\mathsf{Sym}(\mathbb{N})) > \mathfrak{a}_g)$.

$H \leq \mathrm{Sym}(\mathbb{N})$, x a variable. Then W_H is the set of words of the form

$$w = w(x) = g_0 x^{k_0} g_1 \cdots x^{k_l} g_{l+1},$$

where $g_i \in H$, $g_i \neq e$ for $0 < i \leq l$ and $k_i \in \mathbb{Z} \setminus \{0\}$.

$f \in \langle H, h \rangle \setminus H$ then $f = w(h)$ for some $w \in W_H$.

$w \in W_H$, $p \subseteq q$ finite injective functions. Then q is a *good extension* of p w.r.t. w if for all $l \in \mathbb{N}$ s.t.

$$w(p)(l) \text{ is undefined and } w(q)(l) = l$$

there are u, z and n s.t.

$$w = uzu^{-1} \text{ without cancellation}$$
$$u^{-1}(q)(l) = n \text{ and } z(p)(n) = n$$

Domain Extension Lemma:
p finite injective, $k \in \mathbb{N}$, $w \in W_H$ for all but finitely many $n \in \mathbb{N}$ the extension $p \cup \{(k,n)\}$ is a good extension of p w.r.t. w.

Range Extension Lemma:
p finite injective, $n \in \mathbb{N}$, $w \in W_H$ for all but finitely many $k \in \mathbb{N}$ the extension $p \cup \{(k,n)\}$ is a good extension of p w.r.t. w.

Hit f Lemma:
$f \in \mathrm{Sym}(\mathbb{N}) \setminus H$ such that $\langle H, f\rangle$ is cofinitary, p finite injective, $w \in W_H$ for all but finitely many $k \in \mathbb{N}$ the extension $p \cup \{(k, f(k))\}$ is a good extension of p w.r.t. w.

Thm: CH $\Rightarrow$ an MCG exists.

Enumerate Sym($\mathbb{N}$) as $\langle f_\alpha : \alpha < \omega_1 \rangle$.

Recursively construct $\langle g_\alpha : \alpha < \omega_1 \rangle$.

At step $\delta < \omega_1$ have $\langle g_\alpha : \alpha < \delta \rangle$ generating a cofinitary group, construct g_δ by finite approximations.

At step $s < \omega$: only take good extensions w.r.t. $\tilde{w}_0, \ldots, \tilde{w}_s$.

- Hit f_δ if possible.

- Extend Domain.

- Extend Range.

Remember we want: $Con(\mathfrak{a}_g < c(\mathsf{Sym}(\mathbb{N})))$.

Known: In the Miller model:
$c(\mathsf{Sym}(\mathbb{N})) = \aleph_2$
$\Diamond({}^{\mathbb{N}}\mathbb{N}, =^{\infty})$ holds.

Remains:
$\Diamond({}^{\mathbb{N}}\mathbb{N}, =^{\infty}) \Rightarrow \mathfrak{a}_g = \aleph_1$

$\Diamond({}^{\mathbb{N}}\mathbb{N}, =^{\infty})$: For every Borel map $F : {}^{<\omega_1}2 \to {}^{\mathbb{N}}\mathbb{N}$ there is a $G : \omega_1 \to {}^{\mathbb{N}}\mathbb{N}$ s.t. for every $Q : \omega_1 \to 2$ the set

$$\{\alpha \in \omega_1 : F(Q \restriction \alpha) =^{\infty} G(\alpha)\}$$

is stationary.

$F : {}^{<\omega_1}2 \to A$ is Borel if for every δ the restriction of F to ${}^{\delta}2$ is Borel.

$f =^{\infty} g$ for $f, g \in BS$ iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is infinite.

Lem: $\Diamond({}^{\mathbb{N}}\mathbb{N}, =^\infty) \Rightarrow \mathfrak{a}_g = \aleph_1$.

Outline is very similar to CH construction.

By recursion construct $\langle g_\alpha : \alpha < \omega_1 \rangle$. If we have constructed $\langle g_\alpha : \alpha < \delta \rangle$ we construct g_δ by finite approximations.

There are again the steps for extending the domain and range, so that we end up with a permutation of $\mathbb{N}$. Now making it maximal needs a different method.

Have: $\langle g_\alpha : \alpha < \delta \rangle$
Want: g_δ constructed by finite approx, s.t. result is maximal.

$\diamondsuit({}^{\mathbb{N}}\mathbb{N}, =^\infty)$: For every Borel map $F : {}^{<\omega_1}2 \to {}^{\mathbb{N}}\mathbb{N}$ there is a $G : \omega_1 \to {}^{\mathbb{N}}\mathbb{N}$ s.t. for every $Q : \omega_1 \to 2$ the set

$$\{\alpha \in \omega_1 : F(Q \restriction \alpha) =^\infty G(\alpha)\}$$

is stationary.

By coding make the map F take as inputs $(\langle g_\alpha : \alpha < \delta \rangle, g)$ initial parts of construction, and candidate for adding.

Then need if $F(\langle g_\alpha : \alpha < \delta \rangle, g) =^\infty G(\delta)$ then using $G(\delta)$ in the construction results in maximal group.

Have: $\langle g_\alpha : \alpha < \delta\rangle$
Want: g_δ constructed by finite approx, s.t. result is maximal.

Maximality proof:
Suppose $g \notin \langle\{g_\alpha : \alpha < \omega_1\}\rangle$ and $\langle\{g_\alpha : \alpha < \delta\}, g\rangle$ is cofinitary.

Then

$$\{\delta < \omega_1 : F(\langle g_\alpha : \alpha < \delta\rangle, g) =^\infty G(\delta)\}$$

is stationary.

Let δ be a member of this set. Then $F(\langle g_\alpha : \alpha < \delta\rangle, g) =^\infty G(\delta)$.

This means:
for infinitely many n $G(\delta)(n)$ is a correct guess. But then will have by construction g_δ hits g infinitely often.

Have: $\langle g_\alpha : \alpha < \delta \rangle$
Want: g_δ constructed by finite approx, s.t. result is maximal.
Enough: if $F(\langle g_\alpha : \alpha < \delta \rangle, g) =^\infty G(\delta)$, then using $G(\delta)(n)$ infinitely often results in hitting g infinitely often.

Lem: Let H be a cofinitary group, $f \in \mathrm{Sym}(\mathbb{N}) \setminus H$ such that $\langle H, f \rangle$ is cofinitary, and $w \in W_H$. Then for every $k \in \mathbb{N}$ there exists a finite set $S \subseteq f$ such that for every finite injective map p with $|p| \leq k$ there exists a pair $(a, b) \in S$ such that $p \cup \{(a, b)\}$ is a good extension of p w.r.t. w.

Have: $\langle g_\alpha : \alpha < \delta \rangle$
Want: g_δ constructed by finite approx, s.t. result is maximal.
Enough: if $F(\langle g_\alpha : \alpha < \delta \rangle, g) =^\infty G(\delta)$, then using $G(\delta)(n)$ infinitely often results in hitting g infinitely often.

$F(\langle g_\alpha : \alpha < \delta \rangle, g)(n)$ = a code for a finite subset of g s.t. for every finite partial map p of size $\leq 3n$ there is a pair (k, o) encoded s.t. $p \cup \{(k, o)\}$ is a good extension of p w.r.t. $\tilde{w}_0, \ldots, \tilde{w}_n$.

If $\langle \{g_\alpha : \alpha < \delta\}, g \rangle$ is a cofinitary group such a code exists. Otherwise set equal to 0.

Let G be a $\Diamond({}^{\mathbb{N}}\mathbb{N}, =^\infty)$-sequence for this F.

Have: $\langle g_\alpha : \alpha < \delta \rangle$
Want: g_δ constructed by finite approx, s.t. result is maximal.
Enough: if $F(\langle g_\alpha : \alpha < \delta \rangle, g) =^\infty G(\delta)$, then using $G(\delta)(n)$ infinitely often results in hitting g infinitely often.

At step $s < \omega$ in constructing g_δ take good extensions w.r.t. $\tilde{w}_0, \ldots, \tilde{w}_s$.

- If $G(\delta)(n)$ is a valid guess for $\langle g_\alpha : \alpha < \delta \rangle$, then add one of the encoded pairs.

- Extend domain,

- Extend range.

Remember we want: $Con(\mathfrak{a}_g < c(\text{Sym}(\mathbb{N})))$.

Know: In the Miller model:
$c(\text{Sym}(\mathbb{N})) = \aleph_2$
$\Diamond({}^{\mathbb{N}}\mathbb{N}, =^{\infty})$ holds.

Now we also know:
$\Diamond({}^{\mathbb{N}}\mathbb{N}, =^{\infty}) \Rightarrow \mathfrak{a}_g = \aleph_1$

Thm: In the Miller model $\mathfrak{a}_g = \aleph_1 < \aleph_2 = c(\text{Sym}(\mathbb{N}))$.

$\mathfrak{a}_p := \min\{|H| : H \subseteq \mathrm{Sym}(\mathbb{N})$ is a maximal almost disjoint set of permutations $\}$.

Q: Is it consistent with ZFC that $\mathfrak{a}_g$ and $\mathfrak{a}_p$ are different?

Q: $\mathsf{ZFC} \vdash \mathfrak{a} \leq \mathfrak{a}_g, \mathfrak{a}_p$?

Q: Is there a natural cardinal invariant (other than $\mathfrak{c}$) that is an upper bound for $\mathfrak{a}_p$ and $\mathfrak{a}_g$?

Other Questions on Maximal Cofinitary Groups

$G \leq \mathrm{Sym}(\mathbb{N})$ has a natural action on $\mathbb{N}$. We know

Thm: Any maximal cofinitary group has only finitely many orbits.

Q: If G acts on $\mathbb{N}^n$ by diagonal action, does it have only finitely many orbits?

Q: What are the isomorphism types of (maximal) cofinitary groups?

Thm: There exists a locally finite maximal cofinitary group.

Thm: There exists a c.c.c. forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models$ "$\bigoplus_{\alpha<\aleph_1} \mathbb{Z}_2$ has a cofinitary action".

This forcing is canonically obtainable from the group, which leads to: call a group *forcing c.c.c.* iff this associated forcing notion is c.c.c. (same for say proper).

Q: which groups are forcing c.c.c. / proper?