Very MAD Families

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We will work in Baire space ${}^{\mathbb{N}}\mathbb{N}$, the space of all functions $\mathbb{N} \to \mathbb{N}$.

Question 1 (Yi Zhang) Does there exist a mad family (in Baire space) which is Σ^1_1?

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- $f_1, f_2 \in {}^{\mathbb{N}}\mathbb{N}$ are *almost disjoint* iff $\|f_1 \cap f_2\| = \|\{n \mid f_1(n) = f_2(n)\}\|$ is finite.

- $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *maximal almost disjoint* (mad) iff any two members are almost disjoint and it is maximal with this property.

Question 1 Does there exist a mad family (in Baire space) which is Σ^1_1?

Partial Answer (Juris Steprans) There does not exist a strongly mad family which is Σ^1_1.

Juris Steprans There does not exist a strongly mad family which is $\mathbf{\Sigma}^1_1$

- $f_1, f_2 \in {}^{\mathbb{N}}\mathbb{N}$ are *almost disjoint* iff $f_1 \cap f_2$ is finite.

- $f \in {}^{\mathbb{N}}\mathbb{N}$ is *not finitely covered* by $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff $\forall n\ \forall g_1, \ldots, g_n \in \mathcal{A}\ f \setminus \bigcup_i g_i$ is infinite.

- $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *not finitely covered* by $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff all $f \in F$ are not finitely covered by $\mathcal{A}$.

- $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *strongly mad* iff $\mathcal{A}$ is an almost disjoint family and for every countable $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ which is not finitely covered by $\mathcal{A}$ there exists $g \in \mathcal{A}$ such that for all $f \in F$ $f \cap g$ is infinite.

Question 2 Does there exist a strongly mad family which is Π^1_1?

Question 3 When do strongly mad families exist?

- $f_1, f_2 \in {}^{\mathbb{N}}\mathbb{N}$ are *almost disjoint* iff $f_1 \cap f_2$ is finite.

- $f \in {}^{\mathbb{N}}\mathbb{N}$ is *not finitely covered* by $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff $\forall n\ \forall g_1, \ldots, g_n \in \mathcal{A}\ f \setminus \bigcup_i g_i$ is infinite.

- $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *not finitely covered* by $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ iff all $f \in F$ are not finitely covered by $\mathcal{A}$.

- $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *strongly mad* iff $\mathcal{A}$ is an almost disjoint family and for every countable $F \subseteq {}^{\mathbb{N}}\mathbb{N}$ which is not finitely covered by $\mathcal{A}$ there exists $g \in \mathcal{A}$ such that for all $f \in F$ $f \cap g$ is infinite.

- $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ is *very mad* iff $\mathcal{A}$ is an almost disjoint family and for every $F \subseteq {}^{\mathbb{N}}\mathbb{N}$, $\underline{\|F\| < \|\mathcal{A}\|}$ which is not finitely covered by $\mathcal{A}$ there exists $g \in \mathcal{A}$ such that for all $f \in F$ $f \cap g$ is infinite.

Theorem 1 Let M model ZFC + CH. Then there exists a very mad family $\mathcal{A} \in M$ such that

$$M^{\text{Cohen}} \models \check{\mathcal{A}} \text{ is very mad.}$$

Theorem 2 Let M model ZFC and in M κ is a regular cardinal such that $\aleph_1 \leq \kappa \leq 2^{\aleph_0} = \lambda$, then there exists a c.c.c. forcing $\mathbb{P}$ such that $M^{\mathbb{P}}$ satisfies:

- $\lambda = 2^{\aleph_0}$.
- There exists a very mad family $\mathcal{A}$ of cardinality κ.

For $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ define the c.c.c. poset $\mathbb{P}_{\mathcal{A}}$:

elements $\langle s, H \rangle$:
$s : \mathbb{N} \rightharpoonup \mathbb{N}$ finite and partial & $H \subseteq \mathcal{A}$ finite.

order $\langle s_2, H_2 \rangle \leq \langle s_1, H_1 \rangle$:
$s_1 \subseteq s_2$ & $H_1 \subseteq H_2$ & $\forall g \in H_1 \;\; g \cap s_2 \subseteq s_1$.

$\mathbb{P}$ (poset for the theorem):

FS iteration, length κ, of $\mathbb{P}_{\mathcal{A}}$ where $\mathcal{A}$ consists of the generics added so far.

$\mathbb{P}$ is c.c.c. and of cardinality κ.

Lemma If $h : \mathbb{N} \to \mathbb{N} \in N$ (N a model of ZFC), $\mathcal{A} \subseteq {}^{\mathbb{N}}\mathbb{N}$ $\mathcal{A} \in N$, then the generic function G obtained from the forcing by $\mathbb{P}_{\mathcal{A}}$ satisfies

- $\mathcal{A} \cup \{g\}$ is an a.d. family.

- if h is not finitely covered by $\mathcal{A}$, then $h \cap g$ is infinite.

Question 4 Complexity of (very) mad families in Baire space?