

Cofinitary Groups

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- Definitions and basics.
- Definability.
- Cardinalities.
- Orbits.

Definition $\text{Sym}(\mathbb{N})$ is the group of bijections $\mathbb{N} \rightarrow \mathbb{N}$ with group operation composition.

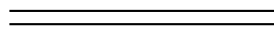
$f \in \text{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \text{Sym}(\mathbb{N})$ is a *cofinitary group* (sharp group) iff all $g \in G$ are cofinitary.

$G \leq \text{Sym}(\mathbb{N})$ is a *maximal cofinitary group (MCG)* iff G is a cofinitary group and is not properly contained in another cofinitary group.

Definition $f \in \text{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \text{Sym}(\mathbb{N})$ is a *cofinitary group* iff all $g \in G$ are cofinitary.



Definition $f, g \in \text{Sym}(\mathbb{N})$ are *almost disjoint* (also called *eventually different*) iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite.

- A group G is cofinitary iff for all $f, g \in G$ f and g are almost disjoint.

- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.
- Any cofinitary group is contained in a maximal cofinitary group.

Question How definable can a maximal cofinitary group be?

Theorem [S. Gao, Y. Zhang] The axiom of constructibility implies the existence of a maximal cofinitary group with a Π_1^1 generating set.

Theorem [BK] The axiom of constructibility implies the existence of a Π_1^1 maximal cofinitary group.

Theorem [O. Spinas] There does not exist a maximal cofinitary group that is locally compact.

Theorem [BK] No maximal cofinitary group is contained in a K_σ .

Conjecture There does not exist a Borel maximal cofinitary group.

Definition Let \mathfrak{a}_g be the least cardinality of a maximal cofinitary group.

Theorem [Y. Zhang] If M is a model of ZFC in which κ is a regular cardinal, $\aleph_0 < \kappa \leq 2^{\aleph_0} = \lambda$, then there is a c.c.c. forcing extension in which $\mathfrak{a}_g = \kappa$ and $2^{\aleph_0} = \lambda$.

Theorem [J. Brendle, O. Spinas, Y. Zhang] It is consistent that $\mathfrak{a} < \mathfrak{a}_p = \mathfrak{a}_g$.

Theorem [BK and Y. Zhang] It is consistent that $c(\text{Sym}(\mathbb{N})) < \mathfrak{a}_p = \mathfrak{a}_g$.

Theorem [BK and T. Hyttinen] It is consistent that \mathfrak{a}_g is bigger than all cardinals in Chicoń's diagram.

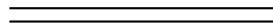
Question Is it consistent that $\mathfrak{a}_g \neq \mathfrak{a}_p$?

Question Is it consistent that \mathfrak{a}_g is singular?

$\text{Sym}(\mathbb{N})$ has a natural action on \mathbb{N} : $(g, n) \mapsto g(n)$.

Theorem [BK] A maximal cofinitary group has finitely many orbits.

Theorem [BK] Martin's axiom implies that for every $n \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{0\}$ there exists a maximal cofinitary group with n finite orbits and m infinite orbits.



$\text{Sym}(\mathbb{N})$ also acts on \mathbb{N}^k for any $k > 1$:
 $(g, n_0, \dots, n_{k-1}) \mapsto (g(n_0), \dots, g(n_{k-1}))$.

Question What is the orbit structure of maximal cofinitary groups acting on \mathbb{N}^k for $k > 1$?

References

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