

# Cofinitary Groups

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- Definitions and basics.
- Definability.
- Cardinalities.
- Orbits.

**Definition**  $\text{Sym}(\mathbb{N})$  is the group of bijections  $\mathbb{N} \rightarrow \mathbb{N}$  with group operation composition.

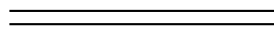
$f \in \text{Sym}(\mathbb{N})$  is *cofinitary* iff  $f$  is the identity or has only finitely many fixed points.

$G \leq \text{Sym}(\mathbb{N})$  is a *cofinitary group* (sharp group) iff all  $g \in G$  are cofinitary.

$G \leq \text{Sym}(\mathbb{N})$  is a *maximal cofinitary group (MCG)* iff  $G$  is a cofinitary group and is not properly contained in another cofinitary group.

**Definition**  $f \in \text{Sym}(\mathbb{N})$  is *cofinitary* iff  $f$  is the identity or has only finitely many fixed points.

$G \leq \text{Sym}(\mathbb{N})$  is a *cofinitary group* iff all  $g \in G$  are cofinitary.



**Definition**  $f, g \in \text{Sym}(\mathbb{N})$  are *almost disjoint* (also called *eventually different*) iff  $\{n \in \mathbb{N} : f(n) = g(n)\}$  is finite.

- A group  $G$  is cofinitary iff for all  $f, g \in G$   $f$  and  $g$  are almost disjoint.

- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size  $|\mathbb{R}|$ .
- Any cofinitary group is contained in a maximal cofinitary group.

**Question** How definable can a maximal cofinitary group be?

**Theorem [S. Gao, Y. Zhang]** The axiom of constructibility implies the existence of a maximal cofinitary group with a  $\Pi_1^1$  generating set.

**Theorem [BK]** The axiom of constructibility implies the existence of a  $\Pi_1^1$  maximal cofinitary group.

**Theorem [O. Spinas]** There does not exist a maximal cofinitary group that is locally compact.

**Theorem [BK]** No maximal cofinitary group is contained in a  $K_\sigma$ .

**Conjecture** There does not exist a Borel maximal cofinitary group.

**Definition** Let  $\mathfrak{a}_g$  be the least cardinality of a maximal cofinitary group.

**Theorem [Y. Zhang]** If  $M$  is a model of ZFC in which  $\kappa$  is a regular cardinal,  $\aleph_0 < \kappa \leq 2^{\aleph_0} = \lambda$ , then there is a c.c.c. forcing extension in which  $\mathfrak{a}_g = \kappa$  and  $2^{\aleph_0} = \lambda$ .

**Theorem [J. Brendle, O. Spinas, Y. Zhang]** It is consistent that  $\mathfrak{a} < \mathfrak{a}_p = \mathfrak{a}_g$ .

**Theorem [BK and Y. Zhang]** It is consistent that  $c(\text{Sym}(\mathbb{N})) < \mathfrak{a}_p = \mathfrak{a}_g$ .

**Theorem [BK and T. Hyttinen]** It is consistent that  $\mathfrak{a}_g$  is bigger than all cardinals in Chicoń's diagram.

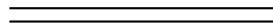
**Question** Is it consistent that  $\mathfrak{a}_g \neq \mathfrak{a}_p$ ?

**Question** Is it consistent that  $\mathfrak{a}_g$  is singular?

$\text{Sym}(\mathbb{N})$  has a natural action on  $\mathbb{N}$ :  $(g, n) \mapsto g(n)$ .

**Theorem [BK]** A maximal cofinitary group has finitely many orbits.

**Theorem [BK]** Martin's axiom implies that for every  $n \in \mathbb{N}$  and  $m \in \mathbb{N} \setminus \{0\}$  there exists a maximal cofinitary group with  $n$  finite orbits and  $m$  infinite orbits.



$\text{Sym}(\mathbb{N})$  also acts on  $\mathbb{N}^k$  for any  $k > 1$ :  
 $(g, n_0, \dots, n_{k-1}) \mapsto (g(n_0), \dots, g(n_{k-1}))$ .

**Question** What is the orbit structure of maximal cofinitary groups acting on  $\mathbb{N}^k$  for  $k > 1$ ?

## References

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