Maximal Cofinitary Groups
how to construct them

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Def: ${}^{\mathbb{N}}\mathbb{N}$ is Baire space, the space of all functions from the natural numbers to the natural numbers.

Sym($\mathbb{N}$) is the symmetry group of natural numbers, the group of all bijections from the natural numbers to the natural numbers.

$g \in$ Sym($\mathbb{N}$) is cofinitary iff it is either the identity or has finitely many fixed points.

$G \subseteq$ Sym($\mathbb{N}$) is a cofinitary group iff it is a subgroup of Sym($\mathbb{N}$) and all it's members are cofinitary.

$G \subseteq$ Sym($\mathbb{N}$) is a maximal cofinitary group iff it is a cofinitary group and not properly contained in any other cofinitary group.

Let $G \subseteq \text{Sym}(\mathbb{N})$ be a group and $g \in \text{Sym}(\mathbb{N}) \setminus G$. What does $\langle G, g \rangle$ look like?

If $h \in \langle G, g \rangle \setminus G$ then $h = w(g)$ for some word $w \in W_G$ where:

Def: W_G is the set of all words of the form

$$g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_l} g_{l+1},$$

with $g_i \in G$, $g_i \neq \text{Id}$ for $0 < i \leq l$ and $k_i \in \mathbb{Z} \setminus \{0\}$ and x some variable.

Def: Let $p, q : \mathbb{N} \rightharpoonup \mathbb{N}$ be finite injective functions with $p \subseteq q$. The q is a *good extension* of p with respect to a $w \in W_G$ iff the following holds:

for any $l \in \mathbb{N}$ such that

$$w(q)(l) = l \text{ and } w(p)(l) \text{ is undefined}$$

there are subwords u and z of w and an $n \in \mathbb{N}$ such that

$$w = u^{-1} z u$$
$$u(q)(l) = n \text{ and } z(p)(n) = n.$$

Let G be a countable cofinitary group, and $n \mapsto w_n$ an enumeration of W_G. Then:

Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f = \bigcup_{s \in \mathbb{N}} p_s$ with p_{s+1} a good extension of p_s with respect to $w_0, \ldots w_s$. Then $\langle G, f \rangle$ is cofinitary.

Lem: (The Domain Extension Lemma)
Let $p : \mathbb{N} \rightharpoonup \mathbb{N}$ be finite and partial and $w \in W_G$. Then for every $n \in \mathbb{N}$ there are at most finitely many $k \in \mathbb{N}$ such that $p \cup \{(n,k)\}$ is *not* a good extension of p with respect to w.

Lem: (The Range Extension Lemma)
Let $p : \mathbb{N} \rightharpoonup \mathbb{N}$ be finite and partial and $w \in W_G$. Then for every $k \in \mathbb{N}$ there are at most finitely many $n \in \mathbb{N}$ such that $p \cup \{(n,k)\}$ is *not* a good extension of p with respect to W.

Shows you can actually construct f as below (from previous slide):

Let G be a countable cofinitary group, and $n \mapsto w_n$ an enumeration of W_G. Then:

Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f = \bigcup_{s \in \mathbb{N}} p_s$ with p_{s+1} a good extension of p_s with respect to $w_0, \ldots w_s$. Then $\langle G, f \rangle$ is cofinitary.

Lem: Let G be cofinitary, $f \in \mathrm{Sym}(N) \setminus G$ such that $\langle G, f\rangle$ is cofinitary. Then if $p : \mathbb{N} \rightharpoonup \mathbb{N}$ is finite partial for every $w \in W_G$ there are only finitely many pairs (a, b) in f such that $p \cup \{(a, b)\}$ is *not* a good extension of p with respect to w.

Construction 1:

G countable cofinitary

$f \in \mathrm{Sym}(\mathbb{N}) \setminus G$

$\langle G, f\rangle$ cofinitary

$\rightsquigarrow$

$h \in \mathrm{Sym}(\mathbb{N}) \setminus G$

$h \cap f$ infinite

$\langle G, h\rangle$ cofinitary

$f \setminus h$ non-empty

h free over G

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Gives more constructive existence of MCG from CH:

Let $\langle f_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all permutations. Construct $\langle g_\alpha : \alpha < \omega_1 \rangle$ a set generating a maximal cofinitary group.

g_α is constructed from the group generated by g_δ, $\delta < \omega_1$ as follows:
at every step do a domain extension, a range extension and an hitting f_α step (if needed).

Thm: (Su Gao and Yi Zhang)
In the constructible universe there exists an MCG G generated by a set $\mathcal{A}$ that is Π^1_1.

Means: In L

$$\mathcal{A} = \{f \in {}^{\mathbb{N}}\mathbb{N} : \phi(f)\},$$

with $\phi \in \Pi^1_1$ (i.e. $\phi(x) \equiv \forall g \in {}^{\mathbb{N}}\mathbb{N}\ \psi(g, x)$ with ψ having only natural number quantifiers, no function quantifiers).

The usual cumulative hierarchy:

$$V_0 = \emptyset \qquad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha, \quad \lambda \text{ limit}$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \qquad V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$$

The constructible hierarchy:

$$L_0 = \emptyset \qquad L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha, \quad \lambda \text{ limit}$$

$$L_{\alpha+1} = \mathrm{Def}(L_\alpha) \qquad L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$$

$\mathrm{Def}(L_\alpha)$: all definable sets: for any formula $\phi(x, \bar{z})$ and parameters $\bar{s} \in L_\alpha$

$$\{x \in L_\alpha : L_\alpha \models \phi(x, \bar{s})\}$$

Have: for all $n \leq \omega$: $L_n = V_n$.

But $V_{\omega+6}$ contains all reals (or slightly later/earlier depending on what a real is to you), whereas all constructible reals are only present at L_{ω_1}.

Lem: If $L_\alpha \cong \mathsf{Sk}(L_\alpha)$ (+technicalities), then there is an $E \subseteq \mathbb{N} \times \mathbb{N}$ such that $E \in L_{\alpha+\omega}$ and $(L_\alpha, \in) \cong (\mathbb{N}, E)$.

Construction 2:

Lem: Let $A = \{g_n : n \in \mathbb{N}\} \subseteq \mathsf{Sym}(\mathbb{N})$ generate a cofinitary group, $E \subseteq \mathbb{N} \times \mathbb{N}$ and $F = \{f_n : n \in \mathbb{N}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ consist of permutations that could be added to the group generated by A.

Then there exists a permutation $g : \mathbb{N} \to \mathbb{N}$ such that $\langle A, g \rangle$ is cofinitary, E is recursive in g and g hits all f_n infinitely often.

Moreover if $A, E, F \in L_\alpha$, then $g \in L_{\alpha+1}$.

Construction: Inductive.

Let $A = \{g_n : n \in \mathbb{N}\}$ be a countable already constructed generating set for a cofinitary group.

Let α be such that $\{g_n : n \in \mathbb{N}\} \in L_\alpha$ and $L_\alpha \cong \mathsf{Sk}(L_\alpha)$.

Let $F \subseteq L_\alpha$ consist of all permutations in L_α that could be added to the cofinitary group generated by A (a countable set).

Let $E \subseteq \mathbb{N} \times \mathbb{N}$ be such that $(L_\alpha, \in) \cong (\mathbb{N}, E)$.

Then use the lemma to construct a g to add to A.

Now $g \in \mathcal{A}$ iff

1. the model encoded in g is wellfounded.

2. $(\mathbb{N}, E)$ encoded in g is isomorphic to some $(L_\alpha, \in)$ and in $(L_{\alpha+\omega}, \in)$ we can see that $g \in \mathcal{A}$.

Need to get our hands on $L_{\alpha+\omega}$. This we can do: there is a Δ^1_0 formula φ such that

$$\varphi(\overbrace{\langle E_\omega, r, u\rangle}^{\text{all coded in a real}}, g) \Leftrightarrow$$

$$(\mathbb{N}, E_\omega) \cong (L_{\alpha+\omega}, \in) \wedge$$

$$r \text{ is the satisfaction relation of } (\mathbb{N}, E_\omega) \wedge$$

$$u \text{ is the element of } (\mathbb{N}, E_\omega) \text{ playing the role of } g$$

We get:

$$
\begin{aligned}
& g \in \mathcal{A} \\
& \quad \Leftrightarrow \\
& \qquad \text{the model encoded in } g \text{ is wellfounded} \wedge \\
& \qquad \forall \langle E_\omega, r, u \rangle \ \Big[\ \varphi(\langle E_\omega, r, u \rangle, g) \wedge \chi(E_\omega, r) \\
& \qquad\qquad\qquad\qquad \rightarrow r(\ulcorner u \in \mathcal{A} \urcorner) = 1 \Big]
\end{aligned}
$$

Here χ takes care of very important details. It ensures E_ω is in fact isomorphic to a level of L.

This shows $\mathcal{A}$ is Π^1_1 in the constructible universe as was desired.

Any group $G \leq \mathrm{Sym}(\mathbb{N})$ has a natural action on the natural numbers. By diagonal action G also has a natural action on $\mathbb{N}^n$ for any $n \in \mathbb{N}$. Sym($\mathbb{N}$) inherits a topology from ${}^{\mathbb{N}}\mathbb{N}$ (the topology determined by initial segments).

Thm: $G \leq \mathrm{Sym}(\mathbb{N})$ is closed iff G it the automorphism group of a countable structure.

Thm: G is the automorphism group of a ω-categorical countable structure iff if is closed and it has only finitely many orbits for the diagonal action on $\mathbb{N}^n$ for every $n \in \mathbb{N}$.

Thm: (BK)
There does not exist an MCG with infinitely many infinite orbits.

Let G be a cofinitary group with infinitely many orbits $\langle \Delta_i : i \in \mathbb{N} \rangle$. Build special h to add.

Then $\langle G, h \rangle$ is cofinitary and $h \notin G$.

Thm: (BK)
Under the continuum hypothesis, for any n and m there is a maximal cofinitary group with n finite and m infinite orbits.

Thm: Assume CH.
An mcg with two infinite orbits exists.

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Construction 3
f can be partial infinite and injective
requirements:
$\forall g \in G \; f \not\subseteq g$ and $\langle G, f \rangle$ "cofinitary".

Construction 4
forget f
have $S \subseteq \mathbb{N}$ infinite and coinfinite

$\rightsquigarrow$

$h \restriction S : S \to S$ is a bijection.

$\mathbb{N} = A \dot{\cup} B$ both infinite

Construct generators for

$$G_0 < \mathrm{Sym}(A) \quad G_1 < \mathrm{Sym}(B)$$

then combine.

I) $f \cap (A \times A)$ infinite.

$(\neg I \Rightarrow)$
II) $f \cap (A \times B)$ infinite
then ran$(f \cap (A \times B))$ infinite
Iterate construction 4:
get something which conjugates to infinite partial injective on A

$$(f \cap (A \times B))^{-1} g^B (f \cap (A \times B))$$

take care of it on A side.

G_0 has generators $\langle g^0_\alpha : \alpha < \omega_1\rangle$
G_1 has generators $\langle g^1_\alpha : \alpha < \omega_1\rangle$

Then $G < \mathrm{Sym}(\mathbb{N})$ generated by

$$g_\alpha(n) = \begin{cases} g^0_\alpha(n) & , n \in A; \\ g^1_\alpha(n) & , n \in B. \end{cases}$$

generates the group we are after.

Thm: (Andreas Blass and BK)
Any cofinitary group contained in a K_σ-set is not maximal.

$G \subseteq \{h \in {}^{\mathbb{N}}\mathbb{N} : h \leq^* f\}$ for some strictly increasing $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) > 0$.

Define from f the sequence of pointed intervals $([i_n, i_{n+1}), p_n)$ with $i_0 := 0$, $p_n := f(i_n)$ and $i_{n+1} := f(p_n)$.

Idea: These pointed intervals are like the orbits in the theorem that no MCG can have infinitely many orbits (more care needed though).

For any g from $g \leq^* f$ for large enough n we have $i_n \leq g(p_n) \leq i_{n+1}$.