A Π^1_1 Maximal Cofinitary Group

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Outline

- Definitions and the Theorem.
- Back of the Proof.
- Constructing Maximal Cofinitary Groups.
- The Coding Lemma.

Definition. Sym($\mathbb{N}$) denotes the group of bijections $\mathbb{N} \to \mathbb{N}$ with operation composition.

$f \in$ Sym($\mathbb{N}$) is *cofinitary* iff it is either the identity or has finitely many fixed points.

$G \leq$ Sym($\mathbb{N}$) is a *cofinitary group* iff all its members are cofinitary.

$G \leq$ Sym($\mathbb{N}$) is a *maximal cofinitary group* iff it is a cofinitary group and is not properly contained in any other cofinitary group.

Theorem (BK). *Under the axiom of constructibility there exists a* Π^1_1 *maximal cofinitary group.*

The Back of The Proof

Assume from now on the axiom of constructibility, $V = L$.

$L = \bigcup_{\alpha \in \mathsf{Ord}} L_\alpha$.

For us everything happens at levels L_α with $\alpha \leq \omega_1$ (every constructible real is in L_{ω_1}).

There is a certain set of levels which have good properties for us. These are the levels where $L_\alpha \cong \mathsf{Sk}(L_\alpha)$:

Definition. For $\alpha > \omega$ we say $L_\alpha \cong \mathsf{Sk}(L_\alpha)$ iff there exists $\langle h, \varphi, \bar{p} \rangle$ (the witness) such that:

- h is a Σ_k-Skolem function, $k \geq 1$, for L_α.

- $h[\mathbb{N} \times (\mathbb{N} \cup \bar{p})] \cong L_\alpha$,

- $h(n, x) = y \Leftrightarrow L_\alpha \models \varphi(\bar{p}, n, x, y)$.

Lemma. *The set* $\{\alpha : L_\alpha \cong \mathsf{Sk}(L_\alpha)\}$ *is unbounded in* ω_1.

Definition. We call $\langle h, \varphi, \bar{p}, r\rangle$ a small witness for $L_\alpha \cong \mathsf{Sk}(L_\alpha)$ if $\langle h, \varphi, \bar{p}\rangle$ is a witness for this, and $\langle h, \varphi, \bar{p}\rangle$ is related to $r \in L_{\alpha+1} \setminus L_\alpha$ as in the proof of the previous lemma.

Let $\langle \beta_\gamma : \gamma < \omega_1\rangle$ increasingly enumerate those β for which L_β is isomorphic to its Skolem hull with small witness.

The reason these levels are nice for us:

Lemma. *If $L_\alpha \cong \mathsf{Sk}(L_\alpha)$, then there is an $E \subseteq \mathbb{N}\times\mathbb{N}$ such that $E \in L_{\alpha+\omega}$ and $(L_\alpha, \in) \cong (\mathbb{N}, E)$.*

Lemma. *There is a formula φ only involving natural number quantifiers such that*

$$\varphi(\langle E_\omega, r\rangle, E) \Leftrightarrow (\mathbb{N}, E_\omega) \cong (L_{\alpha+\omega}, \in) \wedge$$
$$r \text{ is the satisfaction relation for } (\mathbb{N}, E_\omega)$$

Lemma. $\exists$ *a formula* φ *only involving natural number quantifiers s. t.*

$$\varphi(\langle E_\omega, r\rangle, E) \Leftrightarrow (\mathbb{N}, E_\omega) \cong (L_{\alpha+\omega}, \in) \wedge$$
$$r \text{ is the satisfaction relation for } (\mathbb{N}, E_\omega)$$

If $g \in {}^{\mathbb{N}}\mathbb{N} \cap L_{\alpha+\omega}$ is such that E is recursive in g then get from this the following:

Lemma. *There is a formula* φ *only involving natural number quantifiers such that*

$$\varphi(\langle E_\omega, r, u\rangle, g) \Leftrightarrow (\mathbb{N}, E_\omega) \cong (L_{\alpha+\omega}, \in) \wedge$$
$$r \text{ is the satisfaction relation for } (\mathbb{N}, E_\omega) \wedge$$
$$u \in \mathbb{N} \text{ is the element representing } g.$$

If we can build a family $\mathcal{A}$ as $\langle g_\alpha : \alpha < \omega_1 \rangle$ recursively such that:

At step β:

Have build $\langle g_\alpha : \alpha < \beta \rangle$.

Choose γ next in the enumeration of nice levels such that $\langle g_\alpha : \alpha < \beta \rangle \in L_\gamma$.

Have $E \in L_{\gamma+\omega}$ such that $(\mathbb{N}, E) \cong L_\gamma$.

Construct g_β such that E is recursive in g_β and $g_\beta \in L_{\gamma+\omega}$.

Then (almost):

$g \in \mathcal{A} \Leftrightarrow \forall \langle E_\omega, r, u \rangle \; \varphi(\langle E_\omega, r, u \rangle, g) \rightarrow r(\ulcorner u \in \mathcal{A} \urcorner, \bar{\emptyset}) = 1.$

almost: don't know that things encoded are actually levels of the constructible hierarchy.

For wellfounded structures $(\mathbb{N}, E_\omega)$ with satisfaction relation r there is a Σ^1_1-formula $\chi(E_\omega, r)$ such that

$\chi(E_\omega, r) \Leftrightarrow \exists\delta$ a limit ordinal such that $(\mathbb{N}, E_\omega) \cong (L_\delta, \in)$

So . . .

If we can build a family $\mathcal{A}$ as $\langle g_\alpha : \alpha < \omega_1 \rangle$ recursively such that:

At step β:

Have build $\langle g_\alpha : \alpha < \beta \rangle$.

Choose γ next in the enumeration of nice levels such that $\langle g_\alpha : \alpha < \beta \rangle \in L_\gamma$.

Have $E \in L_{\gamma+\omega}$ such that $(\mathbb{N}, E) \cong L_\gamma$.

Construct g_β such that E is recursive in g_β and $g_\beta \in L_{\gamma+\omega}$.

Then:

$g \in \mathcal{A} \Leftrightarrow$
 the model encoded in g is wellfounded $\wedge$
 $\forall \langle E_\omega, r, u \rangle$
 $\{\varphi(\langle E_\omega, r, u \rangle, g) \wedge \chi(E_\omega, r) \rightarrow$
 $r(\ulcorner u \in \mathcal{A} \urcorner, \bar{\emptyset}) = 1\}.$

Constructing Cofinitary Groups

The cofinitary group will be $\langle\{g_\alpha : \alpha < \omega_1\}\rangle$. And the g_α will be constructed recursively.

Assume $G = \langle g_\alpha : \alpha < \beta\rangle$ has been constructed. Then we will construct g_β recursively by finite approximations, $g_\beta = \bigcup_{s\in\mathbb{N}} g_{\beta,s}$. We have to ensure that all group elements in $\langle\{g_\alpha : \alpha < \beta\}, g_\beta\rangle$ are cofinitary.

For any h in this group there is a $w(x) \in G * F(x)$ s.t. $w(g_\beta) = h$.

Study $w(g_\beta)$ for $w(x) \in G * F(x)$.

g_{s+1} is a good extension of g_s if it does not add fixed points it doesn't have to add.

Definition. g_{s+1} is a *good extension* of g_s with respect to $w(x) \in G * F(x)$ iff for all l such that $w(g_{s+1})(l) = l$ and $w(g_s)(l)$ is undefined, there exist u, z, n such that

$$w = u^{-1} z u \quad \text{without cancellation}$$
$$z(g_s)(n) = n \quad \text{and} \quad u(g_{s+1})(l) = n$$

G a countable cofinitary group.

$g_s : \mathbb{N} \rightharpoonup \mathbb{N}$ a finite injective function.

$f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary.

$w \in G * F(x)$.

Lemma (Domain Extension). *For all $n \notin \mathrm{dom}(g_s)$ for all but finitely many $k \in \mathbb{N}$ $g_s \cup \{(n, k)\}$ is a good extension of g_s w.r.t. w.*

Lemma (Range Extension). *For all $k \notin \mathrm{ran}(g_s)$ for all but finitely many $n \in \mathbb{N}$ $g_s \cup \{(n, k)\}$ is a good extension of g_s w.r.t. w.*

Lemma (Hitting f). *For all but finitely many $n \in \mathbb{N}$ the extension $g_s \cup \{(n, f(n))\}$ is a good extension of g_s w.r.t. w.*

An easy result obtained the hard way.
CH implies the existence of an MCG.

Enumerate Sym($\mathbb{N}$) as $\langle f_\alpha : \alpha < \omega_1 \rangle$.

At step β have a countable cofinitary group G to which none of f_α, $\alpha < \beta$, could be added. Add a new generator $g_\beta = \bigcup_{s \in \mathbb{N}} g_{\beta,s}$ such that $\langle G, g \rangle$ is cofinitary and f_β can no longer be added.

Start: $g_{\beta,0} := \emptyset$.

Getting $g_{\beta,s+1}$ from $g_{\beta,s}$ (take good extensions w.r.t. all $w_0, \ldots w_s$ and their subwords):

1. Apply Domain Extension to extend domain.

2. Apply Range Extension to extend range.

3. Hit f_β is possible (not always possible if $\langle G, f_\beta \rangle$ is not cofinitary).

The Coding Lemma

If we can build a family $\mathcal{A}$ as $\langle g_\alpha : \alpha < \omega_1 \rangle$ recursively such that:

At step β:

Have build $\langle g_\alpha : \alpha < \beta \rangle$.

Choose γ next in the enumeration of nice levels such that $\langle g_\alpha : \alpha < \beta \rangle \in L_\gamma$.

Have $E \in L_{\gamma+\omega}$ such that $(\mathbb{N}, E) \cong L_\gamma$.

Construct g_β such that E is recursive in g_β and $g_\beta \in L_{\gamma+\omega}$.

Then:

$$\begin{aligned} & g \in \mathcal{A} \Leftrightarrow \\ & \quad \forall \langle E_\omega, r, u \rangle \\ & \quad \text{the model encoded by } E_\omega \text{ is wellfounded} \wedge \\ & \quad \{\varphi(\langle E_\omega, r, u \rangle, g) \wedge \chi(E_\omega, r) \rightarrow \\ & \qquad\qquad\qquad r(\ulcorner u \in \mathcal{A} \urcorner, \bar{\emptyset}) = 1\}. \end{aligned}$$

Problem:

In constructing a group, adding one element adds countably many other elements as well. In the generator we can encode as Su Gao and Yi Zhang did, but that doesn't help for the others.

We want:

Have G countable cofinitary.

Add a g such that all elements $h \in \langle G, g \rangle \setminus G$ encode some set $E \subseteq \mathbb{N} \times \mathbb{N}$. Then the earlier argument goes through.

The type of coding we will use is as follows:

$f : \mathbb{N} \to \mathbb{N}$ encodes a set E iff there is an m_0 such that for a fixed recursive h:

$f(m_0) = (k, \chi_E(0)) =: m_1$, for some $k \in \mathbb{N}$,
$f(m_1) = (k', \chi_E(1)) =: m_2$ for some $k' \in \mathbb{N}$,
etc.

In this case E is recursive in f.

Note: Before E was uniformly recursive in f, here that will not be the case.

For one word:

Let $w(x) \in G{*}F(x)$, then $w(x) = g_0 x^{l_1} g_1 \cdots x^{l_k} g_k$.

Have g_s. Then $w(g_s)$ is a partial map, so there is a $m_0 \notin \mathrm{dom}(w(g_s))$. We want to ensure that $w(g)(m_0) = (k, \chi_E(0))$ (etc as in the coding explained on previous slide).

A a finite set of points to avoid in all extension activities except for the coding. For every word we are coding in there is one element in this set.

But there is a problem. Examplified by: $w(x) = xw'(x)x^{-1}$.

Fix:

Start with G the countable group generated by h defined below. Let E be the even natural numbers and O the odd numbers. So we get $\mathbb{N} = E \cup O \cup \{0\}$. We define h to be

$$h(n) := \begin{cases} 1 & , n = 0; \\ n - 2 & , n \in E; \\ n + 2 & , n \in O. \end{cases}$$

Note: all members of this G are cofinitary and recursive. Since they are recursive they all appear in $L_{\omega+k}$ for some $k \in \mathbb{N}$.

Then instead of $m_1 = (k, \chi_E(0))$ take $m_1 = h((k, \chi_E(0)))$

Final Construction

$\langle g_\alpha : \alpha < \beta$ already constructed. L_γ a nice level such that $\langle g_\alpha : \alpha < \beta \in L_\gamma$. $F := L_\gamma \cap \mathsf{Sym}(\mathbb{N})$.

To get g_{s+1} from g_s take good extensions w.r.t. all $w_0, \ldots w_s$ and their subwords.

1. Extend domain avoid A.

2. Extend range avoid A.

3. For the first n functions in F hit them if we can while avoiding A..

4. Do coding step for all $w_0, \ldots w_s$.

Su Gao, Yi Zhang, *Definable Sets of Generators in Maximal Cofinitary Groups*, preprint, 2003.

Bart Kastermans, Juris Steprāns, Yi Zhang, *Analytic and Coanalytic Families of Almost Disjoint Functions*, preprint, 2005.