Orbits of Maximal Cofinitary Groups

(part of ongoing thesis work)

Bart Kastermans

University of Michigan

bart@kastermans.nl
http://www.kastermans.nl/bart/

$$g : \mathbb{N} \to \mathbb{N} \quad \Leftrightarrow \quad g \subseteq \mathbb{N} \times \mathbb{N}.$$

Def: Sym($\mathbb{N}$) is the group of bijections from $\mathbb{N}$ to $\mathbb{N}$.

$g \in$ Sym($\mathbb{N}$) is cofinitary iff it has only finitely many fixed points or is the identity.

$G \leq$ Sym($\mathbb{N}$) is cofinitary iff all its members are cofinitary.

$G \leq$ Sym($\mathbb{N}$) is a maximal cofinitary group iff it is cofinitary and not properly contained in another cofinitary group.

Note: $G \leq$ Sym($\mathbb{N}$) is cofinitary iff for every $f, g \in G$, f and g are almost disjoint.

Thm: If a cofinitary group has an infinite number of orbits, then it is not maximal.

Thm: Assume CH.
For any $n, m \in \mathbb{N}$ there exists an mcg with n infinite and m finite orbits.

Thm: Assume CH.
An mcg exists.

Construction 1:

G countable cofinitary
$f \in \mathsf{Sym}(\mathbb{N}) \setminus G$
$\langle G, f \rangle$ cofinitary

$\rightsquigarrow$

$h \in \mathsf{Sym}(\mathbb{N}) \setminus G$
$h \cap f$ infinite
$\langle G, h \rangle$ cofinitary
$f \setminus h$ non-empty
h free over G

Thm: Assume CH.
An mcg with two infinite orbits exists.

<u>Construction 2</u>

f can be partial infinite and injective
requirements:
$\forall g \in G\ f \not\subseteq g$ and $\langle G, f\rangle$ "cofinitary".

<u>Construction 3</u>

forget f
have $S \subseteq \mathbb{N}$ infinite and coinfinite

$\rightsquigarrow$

$h \restriction S : S \to S$ is a bijection.

$\mathbb{N} = A \dot\cup B$ both infinite

Construct generators for

$$G_0 < \text{Sym}(A) \quad G_1 < \text{Sym}(B)$$

then combine.

I) $f \cap (A \times A)$ infinite.

$(\neg I \Rightarrow)$
II) $f \cap (A \times B)$ infinite
then ran$(f \cap (A \times B))$ infinite
Iterate construction 3:
get something which conjugates to infinite partial injective on A

$$(f \cap (A \times B))^{-1} g^B (f \cap (A \times B))$$

take care of it on A side.

G_0 has generators $\langle g_\alpha^0 \mid \alpha < \omega_1 \rangle$
G_1 has generators $\langle g_\alpha^1 \mid \alpha < \omega_1 \rangle$

Then $G < \mathrm{Sym}(\mathbb{N})$ generated by

$$g_\alpha(n) = \begin{cases} g_\alpha^0(n) & , n \in A; \\ g_\alpha^1(n) & , n \in B. \end{cases}$$

generates the group we are after.

References:

Su Gao and Yi Zhang, *Definable sets of generators in maximal cofinitary groups*, preprint.

bart@kastermans.nl
http://www.kastermans.nl/bart/