

Orbits of Maximal Cofinitary Groups

(part of ongoing thesis work)

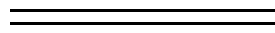
Bart Kastermans

University of Michigan

bart@kastermans.nl

<http://www.kastermans.nl/bart/>

$$g : \mathbb{N} \rightarrow \mathbb{N} \quad \Leftrightarrow \quad g \subseteq \mathbb{N} \times \mathbb{N}.$$

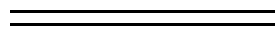


Def: $\text{Sym}(\mathbb{N})$ is the group of bijections from \mathbb{N} to \mathbb{N} .

$g \in \text{Sym}(\mathbb{N})$ is cofinitary iff it has only finitely many fixed points or is the identity.

$G \leq \text{Sym}(\mathbb{N})$ is cofinitary iff all its members are cofinitary.

$G \leq \text{Sym}(\mathbb{N})$ is a maximal cofinitary group iff it is cofinitary and not properly contained in another cofinitary group.



Note: $G \leq \text{Sym}(\mathbb{N})$ is cofinitary iff for every $f, g \in G$, f and g are almost disjoint.

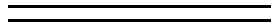
Thm: If a cofinitary group has an infinite number of orbits, then it is not maximal.

Thm: Assume CH.

For any $n, m \in \mathbb{N}$ there exists an mcg with n infinite and m finite orbits.

Thm: Assume CH.

An mcg exists.



Construction 1:

G countable cofinitary

$f \in \text{Sym}(\mathbb{N}) \setminus G$

$\langle G, f \rangle$ cofinitary

\rightsquigarrow

$h \in \text{Sym}(\mathbb{N}) \setminus G$

$h \cap f$ infinite

$\langle G, h \rangle$ cofinitary

$f \setminus h$ non-empty

h free over G

Thm: Assume CH.

An mcg with two infinite orbits exists.

Construction 2

f can be partial infinite and injective
requirements:

$\forall g \in G \ f \not\subseteq g$ and $\langle G, f \rangle$ “cofinitary”.

Construction 3

forget f

have $S \subseteq \mathbb{N}$ infinite and coinfinite

\rightsquigarrow

$h \upharpoonright S : S \rightarrow S$ is a bijection.

$\mathbb{N} = A \dot{\cup} B$ both infinite

Construct generators for

$$G_0 < \text{Sym}(A) \quad G_1 < \text{Sym}(B)$$

then combine.

I) $f \cap (A \times A)$ infinite.

$(\neg I \Rightarrow)$

II) $f \cap (A \times B)$ infinite

then $\text{ran}(f \cap (A \times B))$ infinite

Iterate construction 3:

get something which conjugates to infinite partial injective on A

$$(f \cap (A \times B))^{-1} g^B (f \cap (A \times B))$$

take care of it on A side.

G_0 has generators $\langle g_\alpha^0 \mid \alpha < \omega_1 \rangle$

G_1 has generators $\langle g_\alpha^1 \mid \alpha < \omega_1 \rangle$

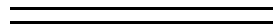
Then $G < \text{Sym}(\mathbb{N})$ generated by

$$g_\alpha(n) = \begin{cases} g_\alpha^0(n) & , n \in A; \\ g_\alpha^1(n) & , n \in B. \end{cases}$$

generates the group we are after.

References:

Su Gao and Yi Zhang, *Definable sets of generators in maximal cofinitary groups*, preprint.



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