A Coanalytic Maximal Cofinitary Group

Bart Kastermans

kasterma@math.wisc.edu

http://www.bartk.nl/

Outline

- Definitions and Basics.
- Question of Concrete Example.
- Results.
- Ideas in the Proof.

Definition Sym($\mathbb{N}$): the group of bijections $\mathbb{N} \to \mathbb{N}$, with group operation composition.

$f \in$ Sym($\mathbb{N}$) is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq$ Sym($\mathbb{N}$) is a *cofinitary group* (sharp group) iff all $g \in G$ are cofinitary.

$G \leq$ Sym($\mathbb{N}$) is a *maximal cofinitary group (MCG)* iff G is a cofinitary group and is not properly contained in another cofinitary group.

Definition $f \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* iff all $g \in G$ are cofinitary.

Definition $f, g \in \mathrm{Sym}(\mathbb{N})$ are *almost disjoint* (also called *eventually different*) iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite.

- A group G is cofinitary iff for all $f, g \in G$ f and g are almost disjoint.

- (Adeleke, Truss) A maximal cofinitary group can not be countable.

- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.

- Any cofinitary group is contained in a maximal cofinitary group.

- (Yi Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is possible that there is an MCG G with $|G| = \kappa$.

Question Does there exist a concrete example of an MCG?

Question How definable can an MCG be?

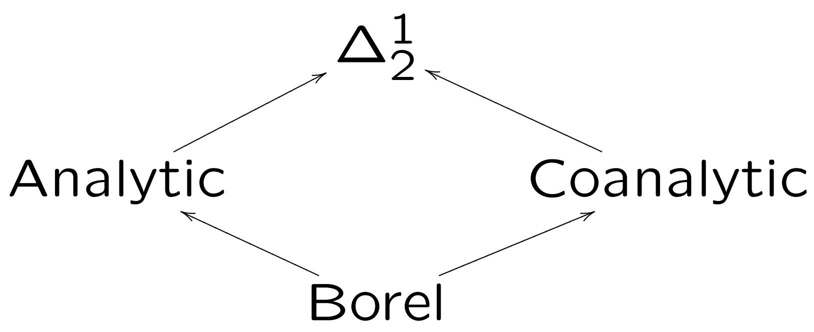
- Borel.
- Topological conditions (closed, compact).
- Projective hierarchy.

Topology on Sym($\mathbb{N}$):
Basic open sets: $\{f \in \text{Sym}(\mathbb{N}) : f \restriction \{0, \ldots, n\} = s\}$ for $s : \{0, \ldots, n\} \to \mathbb{N}$.

Theorem [Otmar Spinas] An MCG can not be locally compact.

Theorem [BK] No MCG is eventually bounded (contained in K_σ).

The question if MCG can be closed is still open (although the conjecture is that they can't be Borel).



Theorem [Andreas Blass] An analytic MCG is Borel.

Conjecture MCG can't be Borel.

Under the Axiom of Constructability, by a standard argument you get an MCG of complexity Δ^1_2.

Theorem [Su Gao and Yi Zhang] Under the Axiom of Constructibility there exists an MCG with a coanalytic generating set.

Theorem [BK] Under the Axiom of Constructibility there exists a coanalytic MCG.

Axiom of constructibility, $V = L$.

$L = \bigcup_{\alpha \in \mathsf{Ord}} L_\alpha$. Here $L_{\alpha+1} = \mathsf{Def}(L_\alpha)$.

For us everything happens at levels L_α with $\alpha \leq \omega_1$ (every constructible real is in L_{ω_1}).

There is a certain set of levels ($L_\alpha \cong \mathsf{Sk}(L_\alpha)$) which have good properties for us.

The properties:

- The set of α such that $L_\alpha \cong \mathsf{Sk}(L_\alpha)$ is unbounded in ω_1;
- It can be checking in $L_{\alpha+\omega}$ that $L_\alpha \cong \mathsf{Sk}(L_\alpha)$;
- If $L_\alpha \cong \mathsf{Sk}(L_\alpha)$ then there exists $E \subseteq \mathbb{N} \times \mathbb{N} \in L_{\alpha+\omega}$ such that $(L_\alpha, \in) \cong (\mathbb{N}, E)$;
- From E can get E_ω such that $(L_{\alpha+\omega}, \in) \cong (\mathbb{N}, E_\omega)$. So if E is recursive in g, then can get E_ω from g.

If we can build a family $\mathcal{A}$ as $\langle g_\alpha : \alpha < \omega_1 \rangle$ recursively such that:

At step β:

Have build $\langle g_\alpha : \alpha < \beta \rangle$.

Choose γ next in the enumeration of nice levels such that $\langle g_\alpha : \alpha < \beta \rangle \in L_\gamma$.

Have $E \in L_{\gamma+\omega}$ such that $(\mathbb{N}, E) \cong L_\gamma$.

Construct g_β such that E is recursive in g_β and $g_\beta \in L_{\gamma+\omega}$.

Then:

$$g \in \mathcal{A} \Leftrightarrow$$
$$\text{the model encoded in } g \text{ is wellfounded} \wedge$$
$$\forall \langle E_\omega, r, u \rangle$$
$$\{\varphi(\langle E_\omega, r, u \rangle, g) \wedge \chi(E_\omega, r) \rightarrow$$
$$r(\ulcorner u \in \mathcal{A} \urcorner, \bar{\emptyset}) = 1\}.$$

Lemma [Coding Lemma — Generic Form] If A is a countable almost disjoint family and $E \subseteq \mathbb{N} \times \mathbb{N}$, we can construct a new member f to adjoin to the family A such that

- E is recursive in f;
- if we iterate the constuction ω_1 many times we construct a maximal almost disjoint family.

Need

- E uniform encoded in f;

- f appears only finitely many levels later in the constructible hierarchy;

Usually all is recursive.

Lemma [Coding Lemma — Adjusted] If G is a countable cofinitary group and $E \subseteq \mathbb{N} \times \mathbb{N}$, we can construct a new member g to adjoin to the family A such that

- E encoded in each $\langle G, g \rangle \setminus G$, uniform only using additional parameter (ϵ, m);

- if we iterate the constuction ω_1 many times we construct an MCG.

Need

- g appears only finitely many levels later in the constructible hierarchy;

All is recursive.

$$
\begin{aligned}
&g \in G \Leftrightarrow \\
&\quad \exists(\epsilon, m)\Big[\text{the model encoded in } g \text{ is wellfounded} \wedge \\
&\quad \forall\langle E_\omega, r, u\rangle \\
&\quad \{\varphi(\langle E_\omega, r, u\rangle, g) \wedge \chi(E_\omega, r) \rightarrow \\
&\qquad\qquad\qquad\qquad r(\ulcorner u \in \mathcal{A}\urcorner, \bar{\emptyset}) = 1\}\Big].
\end{aligned}
$$

The cofinitary group will be $\langle\{g_\alpha : \alpha < \omega_1\}\rangle$. And the g_α will be constructed recursively.

Assume $G = \langle g_\alpha : \alpha < \beta\rangle$ has been constructed. Then we will construct g_β recursively by finite approximations, $g_\beta = \bigcup_{s\in\mathbb{N}} g_{\beta,s}$. We have to ensure that all group elements in $\langle\{g_\alpha : \alpha < \beta\}, g_\beta\rangle$ are cofinitary.

For any h in this group there is a $w(x) \in G * F(x)$ s.t. $w(g_\beta) = h$.

Study $w(g_\beta)$ for $w(x) \in G * F(x)$.

g_{s+1} is a good extension of g_s if it does not add fixed points it doesn't have to add.

Definition g_{s+1} is a *good extension* of g_s with respect to $w(x) \in G * F(x)$ iff for all l such that $w(g_{s+1})(l) = l$ and $w(g_s)(l)$ is undefined, there exist u, z, n such that

$$w = u^{-1}zu \quad \text{without cancellation}$$
$$z(g_s)(n) = n \quad \text{and} \quad u(g_{s+1})(l) = n$$

G a countable cofinitary group.

$g_s : \mathbb{N} \rightharpoonup \mathbb{N}$ a finite injective function.

$f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary.

$w \in G * F(x)$.

Lemma [Domain Extension] For all $n \notin \mathrm{dom}(g_s)$ for all but finitely many $k \in \mathbb{N}$ $g_s \cup \{(n,k)\}$ is a good extension of g_s w.r.t. w.

Lemma [Range Extension] For all $k \notin \mathrm{ran}(g_s)$ for all but finitely many $n \in \mathbb{N}$ $g_s \cup \{(n,k)\}$ is a good extension of g_s w.r.t. w.

Lemma [Hitting f] For all but finitely many $n \in \mathbb{N}$ the extension $g_s \cup \{(n, f(n))\}$ is a good extension of g_s w.r.t. w.

The type of coding we will use is as follows:

$f : \mathbb{N} \to \mathbb{N}$ encodes a set E iff there is an m_0 such that for a fixed recursive h:

$f(m_0) = (k, \chi_E(0)) =: m_1$, for some $k \in \mathbb{N}$,
$f(m_1) = (k', \chi_E(1)) =: m_2$ for some $k' \in \mathbb{N}$,
etc.

In this case E is recursive in f, uniform recursive in (f, m_0).

For one word:

Let $w(x) \in G{*}F(x)$, then $w(x) = g_0 x^{l_1} g_1 \cdots x^{l_k} g_k$.

Have g_s. Then $w(g_s)$ is a partial map, so there is a $m_0 \notin \mathrm{dom}(w(g_s))$. We want to ensure that $w(g)(m_0) = (k, \chi_E(0))$ (etc as in the coding explained on previous slide).

A a finite set of points to avoid in all extension activities except for the coding. For every word we are coding in there is one element in this set.

But there is a problem. Examplified by: $w(x) = xw'(x)x^{-1}$.

Fix:

Start with G the countable group generated by h defined below. Let E be the even natural numbers and O the odd numbers. So we get $\mathbb{N} = E \cup O \cup \{0\}$. We define h to be

$$h(n) := \begin{cases} 1 & , n = 0; \\ n - 2 & , n \in E; \\ n + 2 & , n \in O. \end{cases}$$

Note: all members of this G are cofinitary and recursive. Since they are recursive they all appear in $L_{\omega+k}$ for some $k \in \mathbb{N}$.

Then instead of $m_1 = (k, \chi_E(0))$ take $m_1 = h((k, \chi_E(0)))$

Final Construction

Start with the G from the previous slide.

Then if $\langle g_\alpha : \alpha < \beta \rangle$ is already constructed. L_γ a nice level such that $\langle g_\alpha : \alpha < \beta \rangle \in L_\gamma$. $F := L_\gamma \cap \mathrm{Sym}(\mathbb{N})$.

To get g_{s+1} from g_s take good extensions w.r.t. all $w_0, \ldots w_s$ and their subwords.

1. Extend domain avoid A.

2. Extend range avoid A.

3. For the first s functions in F hit them if we can while avoiding A..

4. Do coding step for all $w_0, \ldots w_s$.

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