# A Locally Finite Maximal Cofinitary Group

Bart Kastermans

Department of Mathematics
University of Wisconsin

kasterma@math.wisc.edu
http://www.bartk.nl/

October 7, 2007: Special Session on Set Theory of the Continuum, Rutgers.

# Outline

- Definitions and Basics.
- Motivation.
- The Result and Idea.

## Definition

Sym($\mathbb{N}$): the group of bijections $\mathbb{N} \to \mathbb{N}$ with operation composition.

$f \in$ Sym($\mathbb{N}$) is *cofinitary* iff $f$ is the identity or has only finitely many fixed points.

$G \leq$ Sym($\mathbb{N}$) is a *cofinitary group* (sharp group) iff all $g \in G$ are cofinitary.

$G \leq$ Sym($\mathbb{N}$) is a *maximal cofinitary group (MCG)* iff $G$ is a cofinitary group and is not properly contained in another cofinitary group.

- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.
- Any cofinitary group is contained in a maximal cofinitary group.
- (Yi Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is possible that there is an MCG $G$ with $|G| = \kappa$.

A cofinitary group is an almost disjoint (eventually different) family of permutations that is also a group.

**Theorem (Su Gao, Yi Zhang)**

*$V = L$ implies there is an mcg with a coanalytic generating set.*

**Theorem**

*$V = L$ implies there is a coanalytic mcg.*

Question
Does there exist an analytic mcg?

### Theorem (Otmar Spinas)

*There does not exist a locally compact mcg.*

### Theorem

*There does not exist an eventually bounded mcg.*

# Constructing MCG

Use CH or MA.

$$G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle = (G_\alpha * F(x))[x := g_\alpha]$$

$$g_\alpha = \bigcup_{s\in\mathbb{N}} g_{\alpha,s}$$

Study

$$w(g_{\alpha,s}) \rightsquigarrow w(g_{\alpha,s+1})$$

*good extension*: no unavoidable new fixed points.

$$w = u^{-1}vu$$

"Always" gives *free groups*.

### Theorem

*MA implies there exists an mcg into which every countable group embeds.*

$$G = *_{\alpha<\mathfrak{c}} G_a.$$

with $G_\alpha$ of all different isomorphism types.

Question

Are the possible complexities of free mcg the same as the possible complexities of all mcg?

**Theorem**

*MA implies there exists a locally finite mcg.*

A finite cg has action on $\mathbb{N}$ with all finite orbits. On all but finitely many of these orbits it acts regularly (transitive and fixed point free).

Any regular action of a group is isomorphic to the group acting on itself.

Have $G = \{g_n : n < \omega\}$ a locally finite cofinitary group.
Let $G_n$ denote $\langle g_i : i \leq n \rangle$.

$f : \mathbb{N} \rightharpoonup \mathbb{N}$ a finite injective map.
Extend $f$ and make act nice with larger and larger parts of $G$.

### Lemma

*Let $G$ be a finite cofinitary group, and $h \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, h \rangle$ is cofinitary. Then for all finite subgroups $H \leq G$ and all but finitely many $n \in \mathbb{N}$ the numbers $n$ and $h(n)$ are not in the same $H$ orbit.*