Maximal Cofinitary Groups

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Outline

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- Results
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- Methods.
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  - Cardinal Characteristics.

Definitions

$\mathrm{Sym}(\mathbb{N})$: the group of bijections $\mathbb{N} \to \mathbb{N}$.

$f \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* (sharp group) iff all $g \in G$ are cofinitary.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *maximal cofinitary group (MCG)* iff G is a cofinitary group and is not properly contained in another cofinitary group.

Definition

$f \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* iff all $g \in G$ are cofinitary.

Definition

$f, g \in \mathrm{Sym}(\mathbb{N})$ are *almost disjoint* (also called *eventually different*) iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite (equivelently iff $f \cap g$ is finite).

- A group G is cofinitary iff for all $f, g \in G$ f and g are almost disjoint.

- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.
- Any cofinitary group is contained in a maximal cofinitary group.

Complexity

Question

How concrete can a maximal cofinitary group be?

Translates to (Boban Velickovic problem list):

Question

What is the least complexity of a formula φ such that $\{g \in \mathrm{Sym}(\mathbb{N}) : \varphi(g)\}$ is a maximal cofinitary group?

Very related to this are topological conditions

Question

Can a MCG be closed in $\mathrm{Sym}(\mathbb{N})$?

Theorem (BK)

It is possible that there exists a coanalytic MCG.

Theorem (Otmar Spinas)

There does not exist a locally compact MCG.

Better lower bounds?

Conjecture

There does not exist a Borel MCG.

Methods

One most often grows groups. We get an MCG G by constructing a sequence $G_\alpha \subseteq G_{\alpha+1}$ such that $G = \bigcup_{\alpha<\kappa} G_\alpha$. And then $G_{\alpha+1} = \langle G_\alpha, g\rangle$.

For this need to understand how to add an element g.

For any $h \in \langle G, g\rangle$ there is a $w(x) \in G * F(x)$ s.t. $w(g) = h$.

Study $w(g)$ for $w(x) \in G * F(x)$.

Grow the g we want to add: $g = \bigcup_{s\in\mathbb{N}} g_s$, g_s finite injective $\mathbb{N} \rightharpoonup \mathbb{N}$, $g_s \subseteq g_{s+1}$.

Grow the g we want to add: $g = \bigcup_{s \in \mathbb{N}} g_s$, g_s finite injective $\mathbb{N} \rightharpoonup \mathbb{N}$, $g_s \subseteq g_{s+1}$.

g_{s+1} is a good extension of g_s if it does not add fixed points it doesn't have to add.

Definition

g_{s+1} is a *good extension* of g_s with respect to $w(x) \in G * F(x)$ iff for all l such that $w(g_{s+1})(l) = l$ and $w(g_s)(l)$ is undefined, there exist u, z, n such that

$$w = u^{-1} z u \quad \text{without cancellation}$$
$$z(g_s)(n) = n \quad \text{and} \quad u(g_{s+1})(l) = n$$

Much of the work on MCG is on figuring out how to go from g_s to g_{s+1}.

G a countable cofinitary group.
$g_s : \mathbb{N} \rightharpoonup \mathbb{N}$ a finite injective function.
$f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary.
$w \in G * F(x)$.

Lemma (Domain Extension)

For all $n \notin \mathrm{dom}(g_s)$ for all but finitely many $k \in \mathbb{N}$ $g_s \cup \{(n, k)\}$ is a good extension of g_s w.r.t. w.

Lemma (Range Extension)

For all $k \notin \mathrm{ran}(g_s)$ for all but finitely many $n \in \mathbb{N}$ $g_s \cup \{(n, k)\}$ is a good extension of g_s w.r.t. w.

Lemma (Hitting f)

For all but finitely many $n \in \mathbb{N}$ the extension $g_s \cup \{(n, f(n))\}$ is a good extension of g_s w.r.t. w.

An easy result obtained the hard way.
CH implies the existence of an MCG.
Enumerate $\mathrm{Sym}(\mathbb{N})$ as $\langle f_\alpha : \alpha < \omega_1 \rangle$.
At step β have a countable cofinitary group G to which none of f_α, $\alpha < \beta$, can be added. Add a new generator $g_\beta = \bigcup_{s \in \mathbb{N}} g_{\beta,s}$ such that $\langle G, g \rangle$ is cofinitary and f_β can no longer be added.
Start: $g_{\beta,0} := \emptyset$.
Getting $g_{\beta,s+1}$ from $g_{\beta,s}$ (take good extensions w.r.t. all $w_0, \ldots w_s$ and their subwords):

1. Apply Domain Extension to extend domain.
2. Apply Range Extension to extend range.
3. Hit f_β if possible.

Orbits and Isomorphism Types

$G \leq \mathrm{Sym}(\mathbb{N})$ has action $(g, n) \mapsto g(n)$.
Any group G that has a faithful action on $\mathbb{N}$ embeds into $\mathrm{Sym}(\mathbb{N})$.

Question

What are the isomorphism types of groups with faithful cofinitary actions?

Question

What is the orbit structure of a MCG acting on $\mathbb{N}$ (or higher powers of $\mathbb{N}$)?

Theorem (BK)

If $G \leq \mathrm{Sym}(\mathbb{N})$ is cofinitary and has infinitely many orbits, then G is not maximal.

Theorem (BK)

It is possible that for any $n, m \in \mathbb{N}$ there exists a MCG with n finite orbits and m infinite orbits.

The question of orbits on $\mathbb{N}^k$, $k > 1$, is still open.

Theorem (BK)

It is possible that $\bigoplus_{\alpha \in \aleph_1} \mathbb{Z}_2$ *has a faithful cofinitary action.*

Question

Which groups are forcing c.c.c.? Which groups are forcing proper?

Cardinal Characteristics

- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.
- Any cofinitary group is contained in a maximal cofinitary group.

Question

(P. Cameron, P. Neumann) If the continuum hypothesis fails, is there an MCG of size $< |\mathbb{R}|$?

(almost nobody believes the continuum hypothesis)

Theorem (Yi Zhang)

If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ *then it is possible that there is an MCG* G *with* $|G| = \kappa$.

Definition

$\mathfrak{a}_g$ is the least cardinality of a maximal cofinitary group.

Corollary

It is possible that $\mathfrak{a}_g < |\mathbb{R}|$.

Theorem (BK, Yi Zhang)

It possible that $\mathfrak{a}_g < c(\mathrm{Sym}(\mathbb{N}))$.

Theorem (Jörg Brendle, Otmar Spinas, Yi Zhang)

$\mathrm{Non}(\mathcal{M}) \leq \mathfrak{a}_g$.

Definition

$\mathfrak{a}_p$ is the least cardinality of a maximal almost disjoint subset of $\mathrm{Sym}(\mathbb{N})$.

Question

Can we prove that $\mathfrak{a}_p = \mathfrak{a}_g$?

References

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