The Question of Concreteness of Cofinitary Subgroups of the Infinite Symmetric Group

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Outline

- Definitions and Basics.
- Question of Concrete Example.
- Results.
- Methods.
- Example Theorem.
- Questions.

Definition

$\mathrm{Sym}(\mathbb{N})$: the group of bijections $\mathbb{N} \to \mathbb{N}$.

$f \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* (sharp group) iff all $g \in G$ are cofinitary.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *maximal cofinitary group (MCG)* iff G is a cofinitary group and is not properly contained in another cofinitary group.

Definition

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$G \leq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* iff all $g \in G$ are cofinitary.

Definition

$f, g \in \mathrm{Sym}(\mathbb{N})$ are *almost disjoint* (also called *eventually different*) iff $\{n \in \mathbb{N} : f(n) = g(n)\}$ is finite.

- A group G is cofinitary iff for all $f, g \in G$ f and g are almost disjoint.

- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.
- Any cofinitary group is contained in a maximal cofinitary group.

Question

(P. Cameron, P. Neumann) If the continuum hypothesis fails, is there an MCG of size $< |\mathbb{R}|$?

(almost nobody believes the continuum hypothesis)

Theorem (Yi Zhang)

If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is possible that there is an MCG G with $|G| = \kappa$.

Question

Does there exist a concrete example of an MCG?

Question

How definable can an MCG be?

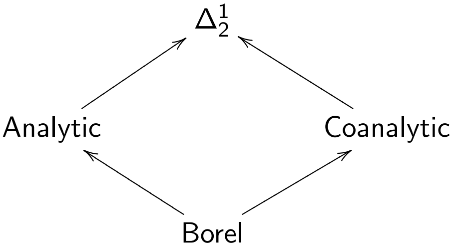
- Borel.
- Topological conditions (closed, compact).
- Projective hierarchy.

Topology on $\mathrm{Sym}(\mathbb{N})$:
Basic open sets: $\{f \in \mathrm{Sym}(\mathbb{N}) : f \upharpoonright \{0, \ldots, n\} = s\}$ for $s : \{0, \ldots, n\} \to \mathbb{N}$.

Definability:
$\mathcal{A} = \{g \in \mathrm{Sym}(\mathbb{N}) : \varphi(g)\}$ with φ a formula.

Complexity of formula gives hierarchy:

- Borel: φ only contains quantifiers $\forall n \in \mathbb{N}$ and $\exists n \in \mathbb{N}$ using a real number as a parameter.
- Analytic (Σ^1_1): $\exists r \in \mathrm{Sym}(\mathbb{N})\psi(r, g)$ with ψ Borel.
- Coanalytic (Π^1_1): $\forall r \in \mathrm{Sym}(\mathbb{N})\psi(r, g)$ with ψ Borel.
- Δ^1_2 is just above analytic and coanalytic: has formula $\forall r_1 \in \mathrm{Sym}(\mathbb{N})\exists r_2 \in \mathrm{Sym}(\mathbb{N})\psi(r_1, r_2, g)$ and $\exists r_1 \in \mathrm{Sym}(\mathbb{N})\forall r_2 \in \mathrm{Sym}(\mathbb{N})\phi(r_1, r_2, g)$.

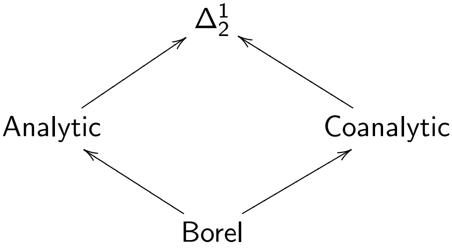


Arrows denote inclusions.

Theorem (Otmar Spinas)

An MCG can not be locally compact.

The question if MCG can be closed is still open (tough the conjecture is that they can't be Borel).



Theorem (Andreas Blass)

If there is an analytic MCG then there is a Borel MCG.

Conjecture

MCG can't be Borel.

Axiom of Constructability.

By a standard argument you get an MCG of complexity Δ^1_2.

Theorem (Su Gao and Yi Zhang)

Under the Axiom of Constructibility there exists an MCG with a coanalytic generating set.

Using some of their ideas:

Theorem (BK)

Under the Axiom of Constructibility there exists a coanalytic MCG.

One most often grows groups. We get an MCG G by constructing a sequence $G_\alpha \subseteq G_{\alpha+1}$ such that $G = \bigcup_{\alpha<\kappa} G_\alpha$. And then $G_{\alpha+1} = \langle G_\alpha, g\rangle$.

For this need to understand how to add an element g.

For any $h \in \langle G, g\rangle$ there is a $w(x) \in G * F(x)$ s.t. $w(g) = h$.

Study $w(g)$ for $w(x) \in G * F(x)$.

Grow the g we want to add: $g = \bigcup_{s\in\mathbb{N}} g_s$, g_s finite injective $\mathbb{N} \rightharpoonup \mathbb{N}$, $g_s \subseteq g_{s+1}$.

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g_{s+1} is a good extension of g_s if it does not add fixed points it doesn't have to add.

Definition

g_{s+1} is a *good extension* of g_s with respect to $w(x) \in G * F(x)$ iff for all l such that $w(g_{s+1})(l) = l$ and $w(g_s)(l)$ is undefined, there exist u, z, n such that

$$w = u^{-1} z u \quad \text{without cancellation}$$
$$z(g_s)(n) = n \quad \text{and} \quad u(g_{s+1})(l) = n$$

Much of the work on MCG is on figuring out how to go from g_s to g_{s+1}.

G a countable cofinitary group.
$g_s : \mathbb{N} \rightharpoonup \mathbb{N}$ a finite injective function.
$f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary.
$w \in G * F(x)$.

Lemma (Domain Extension)

For all $n \notin \mathrm{dom}(g_s)$ for all but finitely many $k \in \mathbb{N}$ $g_s \cup \{(n, k)\}$ is a good extension of g_s w.r.t. w.

Lemma (Range Extension)

For all $k \notin \mathrm{ran}(g_s)$ for all but finitely many $n \in \mathbb{N}$ $g_s \cup \{(n, k)\}$ is a good extension of g_s w.r.t. w.

Lemma (Hitting f)

For all but finitely many $n \in \mathbb{N}$ the extension $g_s \cup \{(n, f(n))\}$ is a good extension of g_s w.r.t. w.

Any group $G \leq \mathrm{Sym}(\mathbb{N})$ has a natural action on $\mathbb{N}$: $(g, n) \mapsto g(n)$.

Question

What does this action look like if G is an MCG?

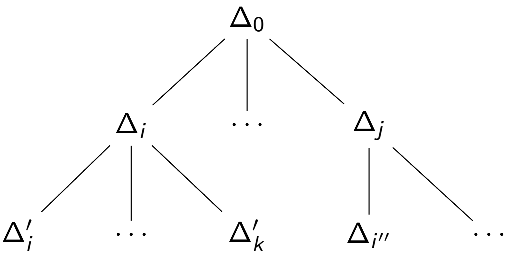
Theorem (BK)

If $G \leq \mathrm{Sym}(\mathbb{N})$ is cofinitary and has infinitely many orbits, then G is not maximal.

Sketch of Proof:
Let Δ_n, $n \in \mathbb{N}$, be the different orbits.

Let $g \in \mathrm{Sym}(\mathbb{N})$ be such that for all n for all $a \neq b \in \Delta_n$ for all $m, m' \in \mathbb{Z}$ (not both 0) we have that

- $g^m(a) \in \Delta_k$, $k \neq n$,
- $g^{m'}(b) \in \Delta_l$, $l \neq n$ and $l \neq k$.



Where Δ_i —— Δ_j means that there is an $a \in \Delta_i$ such that $g(a) \in \Delta_j$ or $g^{-1}(a) \in \Delta_j$. Also we have that for every edge there is a unique pair $(a, g(a))$ or $(a, g^{-1}(a))$ that causes the connection.

Now let $w(x) \in G * F(x)$:

$$w(x) = g_0 x^{l_1} g_1 \cdots x^{l_k} g_k$$
$$w(g) = g_0 g^{l_1} g_1 \cdots g^{l_k} g_k$$

Let $a \in \mathbb{N}$ we'll now see that if a is a fixed point of $w(g)$, we (injectively) find a pair (g_i, n) with n a fixed point for g_i. Which will finish the proof as, under the assumption that $w(g)$ has infinitely many fixed points, we get a $g_i \in G$ with infinitely many fixed points contradicting that g_i is cofinitary.

$$w(g) = g_0 g^{l_1} g_1 \cdots g^{l_k} g_k$$

$$a, g_k(a) \in \Delta_i \longrightarrow gg_k(a) \in \Delta_j \dashrightarrow g_{m'} g^{k_{m'+1}} \cdots g_k(a) \in \Delta_m$$

$$b \in \Delta_i \rightleftarrows g(b), g_i(g(b)) \in \Delta_j$$

Connection unique: Going from $g(b) \in \Delta_j$ *need* to apply g^{-1}. By normal form, first apply a g_i, but this g_i cannot move the element as then g^{-1} doesn't bring it back to Δ_i. ☒

We showed:

Theorem (BK)

If $G \leq \mathrm{Sym}(\mathbb{N})$ *is cofinitary and has infinitely many orbits, then* G *is not maximal.*

But the following is still open:

Question

Can an MCG have infinitely many orbits under its diagonal action on $\mathbb{N} \times \mathbb{N}$ (or $\mathbb{N}^k$, $k \geq 2$) ?

On Boban Velickovic problem list:

Question

Can an MCG be analytic?

Question

What is the lower bound on the complexity of MCG?

Question

What are the isomorphism types of MCG?

Question

Which groups can act cofinitarily?

Peter J. Cameron, *Cofinitary Permutation Groups*, Bull. London Math. Soc. **28** (1996) pp. 113–140.

Peter G. Hinman, *Recursion-Theoretic Hierarchies*, Springer-Verlag, 1978.

Su Gao, Yi Zhang, *Definable Sets of Generators in Maximal Cofinitary Groups*, preprint, 2003.