Orthogonal MAD Families

Bart Kastermans

kasterma@math.wisc.edu
http://www.bartk.nl/

March 16, 2007: AMS Special Session

- Definitions and Background.
- Results and Methods.
- Question.

Let X be a collection of countable objects, two elements $a, b \in X$ are *almost disjoint* iff $a \cap b$ is finite.
A subfamily $\mathcal{A} \subseteq X$ is *almost disjoint* iff for all distinct $a, b \in \mathcal{A}$, $a \cap b$ is finite.
Such a family is maximal almost disjoint iff it is not properly contained in another almost disjoint family.

Examples

Mad families of subsets of $\mathbb{N}$: $X = \mathcal{P}(\mathbb{N})$ (add the requirement that X is infinite).
Mad families of functions: $X = {}^{\mathbb{N}}\mathbb{N}$.

Theorem (Mathias)

There does not exist an analytic mad family of subsets of $\mathbb{N}$.

Question

Does there exist an analytic mad family of functions?

Theorem (Steprāns)

There does not exist an analytic strongly mad family of functions.

Theorem

Under the axiom of constructibility there exists a coanalytic strongly mad family of functions.

---

f *avoids* (is not covered by) $h_0, \ldots, h_n$ iff $f \setminus \bigcup_{i \leq n} h_i$ is infinite.
f *avoids* (is not finitely covered by) $\mathcal{A}$ iff for all $h_0, \ldots, h_n \in \mathcal{A}$, f avoids $h_0, \ldots, h_n$.
F *avoids* $\mathcal{A}$ iff all $f \in F$ avoid $\mathcal{A}$.
$\mathcal{A}$ is strongly (very) mad iff $\mathcal{A}$ is almost disjoint and for any collection F such that F is countable (of cardinality $< |\mathcal{A}|$) and F avoids $\mathcal{A}$ there exists $g \in \mathcal{A}$ such that for all $f \in F$, $f \cap g$ is infinite.

f avoids $h_0, \ldots, h_n$ iff $f \setminus \bigcup_{i \leq n} h_i$ is infinite.
f avoids $\mathcal{A}$ iff for all $h_0, \ldots, h_n \in \mathcal{A}$, f avoids $h_0, \ldots, h_n$.
F avoids $\mathcal{A}$ iff all $f \in F$ avoid $\mathcal{A}$.
$\mathcal{A}$ is orthogonal to $\mathcal{B}$ iff $\mathcal{A}$ avoids $\mathcal{B}$ and $\mathcal{B}$ avoids $\mathcal{A}$.

Note: Applies to both mad families of subsets of $\mathbb{N}$ and to mad families of functions

Theorem (BK)

MA implies there exist mad families of subsets of $\mathbb{N}$ *that are orthogonal.*

Theorem (BK)

MA implies there exists a mad family of subsets of $\mathbb{N}$ *to which no mad family of subsets of* $\mathbb{N}$ *is orthogonal.*

Condition InfHitInf:
$\mathit{InfHitInf}(\mathcal{A}, \mathcal{B})$ iff for all $A \in \mathcal{A}$ there are $\{B_i : i \in \mathbb{N}\} \subseteq \mathcal{B}$ (all different) such that $A \cap B_i$ is infinite for all $i \in \mathbb{N}$.

If for a.d. families $\mathcal{A}$ and $\mathcal{B}$, we have $\mathit{InfHitInf}(\mathcal{A}, \mathcal{B})$, then $\mathcal{A}$ avoids $\mathcal{B}$.

In fact for mad families $\mathcal{A}$ and $\mathcal{B}$: $\mathcal{A}$ is orthogonal to $\mathcal{B}$ iff $\mathit{InfHitInf}(\mathcal{A}, \mathcal{B})$ and $\mathit{InfHitInf}(\mathcal{B}, \mathcal{A})$.

If $\mathcal{A}$ and $S \subseteq \mathbb{N}$ are such that $|\mathcal{A}| < \mathfrak{c}$ and S avoids $\mathcal{A}$, then we can add a subset of S to $\mathcal{A}$.

Theorem (BK)

MA implies there exist mad families of functions that are orthogonal.

Theorem (BK)

MA implies that for every mad family of functions there exists one that is orthogonal to it.

Condition InfHitInf:
InfHitInf$(\mathcal{A}, \mathcal{B})$ iff for all $f \in \mathcal{A}$ there are $\{g_i : i \in \mathbb{N}\} \subseteq \mathcal{B}$ (all different) such that $f \cap g_i$ is infinite for all $i \in \mathbb{N}$.

If for a.d. families $\mathcal{A}$ and $\mathcal{B}$, we have *InfHitInf*$(\mathcal{A}, \mathcal{B})$, then $\mathcal{A}$ avoids $\mathcal{B}$.

InfHitInf gives too many requirements to use MA to construct a family orthogonal to a given one.

Definition

For $\mathcal{B}$ and $\mathcal{A}$ families of functions we call a function $H : [\mathcal{A}]^{<\omega} \to ([\omega]^\omega)^2 \times ([\mathcal{B}]^{<\omega})^2$ *good for* $(\mathcal{A}, \mathcal{B})$ iff for all $f_0, \ldots, f_n \in \mathcal{A}$ we have that $H(\{f_0, \ldots, f_n\})) = \langle W_0, W_1, \{g_1^0, \ldots, g_n^0\}, \{g_1^1, \ldots, g_n^1\}\rangle$ such that

- $f_i \restriction W_0 \subseteq^* g_i^0 \restriction W_0$ for all $i \leq n$,
- $f_i \restriction W_1 \subseteq^* g_i^1 \restriction W_1$ for all $i \leq n$,
- $\{g_1^0, \ldots, g_n^0\} \cap \{g_1^1, \ldots, g_n^1\} = \emptyset$, and
- $W_0 \cap W_1$ is finite.

If such H exist then $\mathcal{A}$ avoids $\mathcal{B}$. Combine with *InfHitInf*$(\mathcal{B}, \mathcal{A})$.

Question

Is it consistent that there exists a mad family of functions to which no other such family is orthogonal?