Generating Sets of Cofinitary Groups

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Outline

- Definitions and Basics.
- Two Questions, their meaning and motivation.
- Some ideas from the proofs.

Definition

Sym($\mathbb{N}$): the group of bijections $\mathbb{N} \to \mathbb{N}$ with operation composition.

$f \in$ Sym($\mathbb{N}$) is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq$ Sym($\mathbb{N}$) is a *cofinitary group* (sharp group) iff all $g \in G$ are cofinitary.

$G \leq$ Sym($\mathbb{N}$) is a *maximal cofinitary group (MCG)* iff G is a cofinitary group and is not properly contained in another cofinitary group.

A cofinitary group is an almost disjoint (eventually different) family of permutations that is also a group.

- (Adeleke, Truss) A maximal cofinitary group can not be countable.
- (P. Neumann) There is a cofinitary group of size $|\mathbb{R}|$.
- Any cofinitary group is contained in a maximal cofinitary group.
- (Yi Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is possible that there is an MCG G with $|G| = \kappa$.

- If a cofinitary group has infinitely many orbits, then it is not maximal.
- Martin's Axiom: any finite number of finite and infinite orbits are possible.
- Martin's Axiom: there exists a locally finite maximal cofinitary group.

Two Questions (Vershik)

Does there exist a dense maximal cofinitary group?

Does there exist a noncomputable cofinitary group with a computable set of generators?

Does there exist a dense maximal cofinitary group?

Topology generated by basic open sets
$[s] = \{f \in \mathrm{Sym}(\mathbb{N}) : s \subseteq f\}$.

Hierarchies above this.

Does there exist a noncomputable cofinitary group with a computable set of generators?

Theorem (Su Gao, Yi Zhang)

$V = L$ implies there is an mcg with a coanalytic generating set.

Theorem (K)

$V = L$ implies there is a coanalytic mcg.

Computable

Does there exist a noncomputable cofinitary group with a computable set of generators?

Computable set of generators:
There is a computer program φ with two inputs such that $\{n \mapsto \varphi(m, n) : m \in \mathbb{N}\}$ is this set of functions.

Computable group $\langle G, \circ \rangle$:
There is a computer program ψ with two inputs such that $(n, m) \mapsto \psi(n, m)$ gives a group structure on $\mathbb{N}$, and this group is isomorphic to $\langle G, \circ \rangle$.

Constructing CG

$$G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle = (G_\alpha * F(x))[x := g_\alpha]$$

$$g_\alpha = \bigcup_{s \in \mathbb{N}} g_{\alpha,s}$$

Study

$$w(g_{\alpha,s}) \rightsquigarrow w(g_{\alpha,s+1})$$

good extension: no unavoidable new fixed points.

$$w = u^{-1} v u$$

G a countable cofinitary group, $f \in \mathrm{Sym}(\mathbb{N}) \setminus G$ such that $\langle G, f \rangle$ is cofinitary, $p : \mathbb{N} \rightharpoonup \mathbb{N}$ finite, and $w \in G * F(x)$.

Lemma (Domain Extension)

For all $n \notin \mathrm{dom}(p)$ *for all but finitely many* k, $p \cup \{(n, k)\}$ *is a good extension of* p *w.r.t.* w.

Lemma (Range Extension)

For all $k \notin \mathrm{ran}(p)$ *for all but finitely many* n, $p \cup \{(n, k)\}$ *is a good extension of* p *w.r.t.* w.

Lemma (Hitting f)

For all but finitely many n, $p \cup \{(n, f(n))\}$ *is a good extension of* p *w.r.t.* w.

Theorem

MA implies there exists an mcg into which every countable group embeds.

$$G = *_{\alpha<\mathfrak{c}} G_a.$$

with G_α of all different isomorphism types.

Does there exist a dense maximal cofinitary group?

Let $\langle s_n : n \in \mathbb{N} \rangle$ be an enumeration all all finite injections $\mathbb{N} \rightharpoonup \mathbb{N}$. Start the different generators with these.

Construction CG with relations

$\bar{R} = \langle R_s \subseteq F(\bar{x}) : s \in \mathbb{N} \rangle$
$G_s = F(\bar{x})/R_s$
$W_{s,\mathrm{Id}} = \{w \in F(\bar{x}) : w =_{G_s} \mathrm{Id}\}$

$\bar{R}$ is a *demure sequence* iff

- for all $t \in \mathbb{N}$, the set $R_s \upharpoonright t$ is finite.
- for all $t \in \mathbb{N}$, $R_{s+1} \upharpoonright t \subseteq R_s \upharpoonright t$.
- for all $t \in \mathbb{N}$, $W_{s,\mathrm{Id}} \upharpoonright t = (R_s \upharpoonright t)^{F(\bar{x} \upharpoonright t)}$.

G_ω is the inverse limit of the groups G_s and $G_\omega = F(\bar{x})/W_{\omega,\mathrm{Id}}$ where $W_{\omega,\mathrm{Id}} = \cap W_{s,\mathrm{Id}}$

Need the isomorphism type of G_ω to be nonrecursive:
Let $A \in \Delta^0_1$ not r.e.
Ensure $n \in A$ iff $\exists l \in \mathbb{N}, g \in G \;\; g^{f(n,l)} = \mathrm{Id}$ (f computable).

$n \in A$ iff $\lim_{l \to \infty} h(n,l) = 1$ (and for all n this limit exists).

$R_s = \{x_i^{p_{n,l}} : \forall j (l \leq j \leq s \;\; h(n,j) = 1)\}$

s-applying relations

$\bar{p} \mapsto \bar{q}$ where $(a, b) \in q_i$ iff there is a $x_i w' \in W_{s,\mathrm{Id}}$ such that $w'(b) = a$.

Theorem (K)

There exists a noncomputable cofinitary group that has a computable set of generators.