Maximal Cofinitary Groups

Bart Kastermans

Department of Mathematics
University of Wisconsin

kasterma@math.wisc.edu
http://www.bartk.nl/

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Boise State University.

Outline

$\mathrm{Sym}(\mathbb{N})$: the group of bijections $\mathbb{N} \to \mathbb{N}$ (permutations) with operation composition.

$f \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* iff all $g \in G$ are cofinitary.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *maximal cofinitary group* iff G is a cofinitary group and is not properly contained in another cofinitary group.

Permutations and operation VS Underlying sets.

(Adeleke, Truss) A maximal cofinitary group can not be countable.

(Neumann) There is a cofinitary group of size $|\mathbb{R}|$.

Any cofinitary group is contained in a maximal cofinitary group.

(Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is consistent that there is a maximal cofinitary group G with $|G| = \kappa$.

A cofinitary group is an almost disjoint (eventually different) family of permutations that is also a group.

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Does there exist a concrete example of a maximal cofinitary group?

(Underlying set)

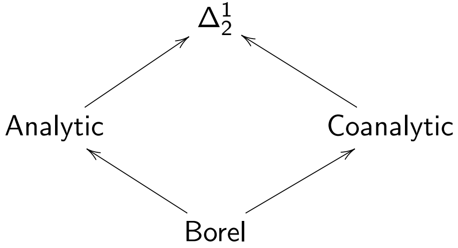
How definable can a maximal cofinitary group be?

- Topological conditions (closed, compact).
- Borel hierarchy.
- Projective hierarchy.

Topology on $\mathrm{Sym}(\mathbb{N})$:

Basic open sets: $\{f \in \mathrm{Sym}(\mathbb{N}) : f \restriction \{0, \ldots, n\} = s\}$ for $s : \{0, \ldots, n\} \to \mathbb{N}$.

Bottom of Projective Hierarchy



Axiom of Constructibility.

Standard argument: the Axiom of Constructibility gives a maximal cofinitary group of complexity Δ^1_2.

Theorem (Gao and Zhang)

Under the Axiom of Constructibility there exists a maximal cofinitary group with a coanalytic generating set.

Theorem (K)

Under the Axiom of Constructibility there exists a coanalytic maximal cofinitary group.

Constructing MCG

Grow group step by step $G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle$.

Grow permutation step by step $g_\alpha = \bigcup_{s \in \mathbb{N}} g_{\alpha,s}$.

Description of larger group in terms of smaller:

$$\langle G, g \rangle = (G * F(x))[x := g]$$

$$w(x) \in G * F(x) \qquad w(x) = g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_m} g_{m+1}$$

Study $w(g_{\alpha,s}) \rightsquigarrow w(g_{\alpha,s+1})$.

good extension: no unavoidable new fixed points.

Gives *free* groups.

Theorem (K)

Martin's Axiom implies that there exists a maximal cofinitary group into which any countable group embeds.

Theorem (K)

Martin's Axiom implies that there exists a locally finite maximal cofinitary group.

$G \leq \mathrm{Sym}(\mathbb{N})$ has action $(f, n) \mapsto f(n)$ on $\mathbb{N}$.

Theorem (K)

Every maximal cofinitary group has finitely many orbits.

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Martin's Axiom implies that for any $n, m \in \mathbb{N}$ with $m \neq 0$ there exists a maximal cofinitary group with n finite orbits and m infinite orbits.

Theorem

Every maximal cofinitary group has finitely many orbits.

Suppose G cofinitary with infinitely many orbits $\langle \Delta_i : i \in \mathbb{N} \rangle$.

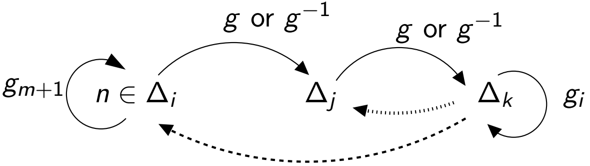
Construct $g \notin G$ such that $\langle G, g \rangle$ is cofinitary.

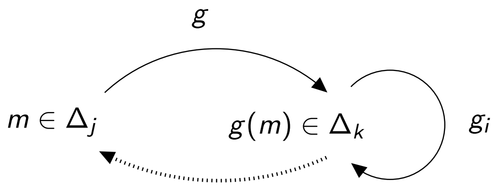
$$\langle G, g \rangle = (G * F(x))[x := g]$$

$$w \in G * F(x) \qquad w = g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_m} g_{m+1}$$

$w(x) = g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_m} g_{m+1}$

From $w(g)(n) = n$ find g_i occuring in $w(x)$ with a fixed point (sufficiently injectively).




g
$m \in \Delta_j$
$g(m) \in \Delta_k$
g_i

Theorem

Every maximal cofinitary group has finitely many orbits.

(Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is consistent that there is a maximal cofinitary group G with $|G| = \kappa$.

$\mathfrak{a}_{\mathfrak{p}}$ the minimal cardinality of a maximal almost disjoint family of permutations.

$\mathfrak{a}_g$ the minimal cardinality of a maximal cofinitary group.

$\mathfrak{a}_{\mathfrak{p}} = \mathfrak{a}_g$?