Maximal Cofinitary Groups

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Outline

Definitions

$\mathrm{Sym}(\mathbb{N})$: the group of bijections $\mathbb{N} \to \mathbb{N}$ (permutations) with operation composition.

$f \in \mathrm{Sym}(\mathbb{N})$ is *cofinitary* iff f is the identity or has only finitely many fixed points.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *cofinitary group* iff all $g \in G$ are cofinitary.

$G \leq \mathrm{Sym}(\mathbb{N})$ is a *maximal cofinitary group* iff G is a cofinitary group and is not properly contained in another cofinitary group.

Some basic properties

Any cofinitary group is contained in a maximal cofinitary group.

(Adeleke, Truss) A maximal cofinitary group cannot be countable.

(Koppelberg) There is a cofinitary group of size $|\mathbb{R}|$.

(Zhang) If $|\mathbb{N}| < \kappa \leq |\mathbb{R}|$ then it is consistent that there is a maximal cofinitary group G with $|G| = \kappa$.

A key property

A group is a cofinitary group iff it is an almost disjoint family.

Definition: $A, B \subseteq \mathbb{N}$ are *almost disjoint* iff $A \cap B$ is finite.

For any $f : \mathbb{N} \to \mathbb{N}$ its graph, $\text{graph}(f) = \{(n, f(n)) : n \in \mathbb{N}\}$, is a subset of $\mathbb{N} \times \mathbb{N}$.

$$f(n) = g(n) \quad \Leftrightarrow$$
$$g^{-1}f(n) = n \quad \Leftrightarrow$$
$$(n, f(n)) \in \text{graph}(f) \cap \text{graph}(g).$$

Main questions

Question: Does there exist a concrete example of a maximal cofinitary group?

Question: What are the possible isomorphism types of maximal cofinitary groups?

ZFC or not

ZFC is standard set theory with axiom of choice.

Objects obtained using the axiom of choice are often very complicated.

Axiom of Constructibility: all sets are constructible.

Martin's Axiom: certain forcing constructions can be completed in the model.

Let $G \leq \mathrm{Sym}(\mathbb{N})$.
G acts on $\mathbb{N}$ by function application: $(f, n) \mapsto f(n)$.

Theorem (Kastermans)

Every maximal cofinitary group has finitely many orbits.

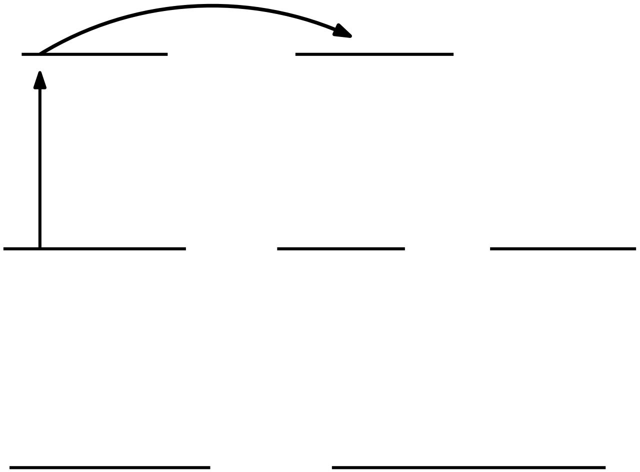
Theorem (Kastermans)

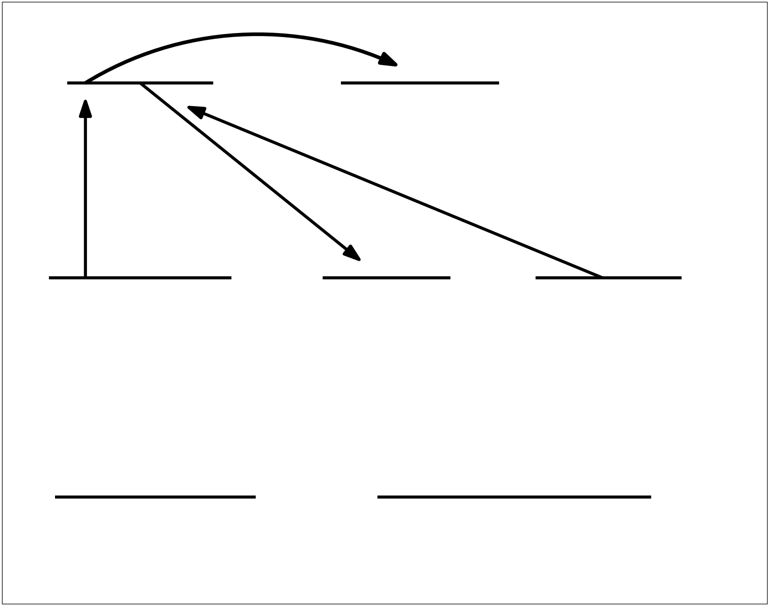
Martin's Axiom implies that for any $n, m \in \mathbb{N}$ with $m \neq 0$ there exists a maximal cofinitary group with n finite orbits and m infinite orbits.

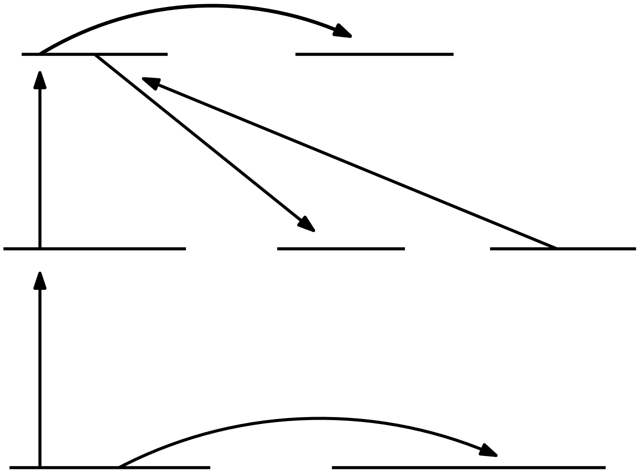
Towards a proof of the first theorem:

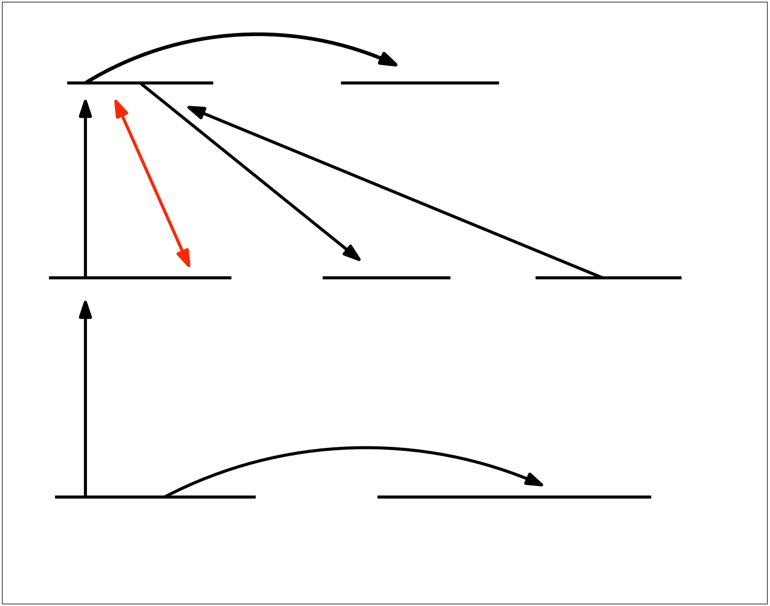
Suppose G cofinitary with infinitely many orbits $\langle \Delta_i : i \in \mathbb{N} \rangle$.

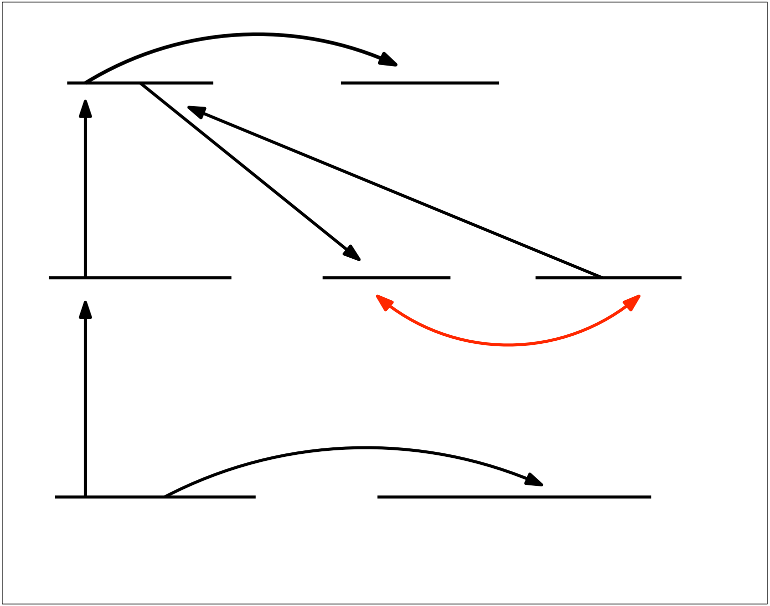
Construct $g \notin G$ such that $\langle G, g \rangle$ is cofinitary.





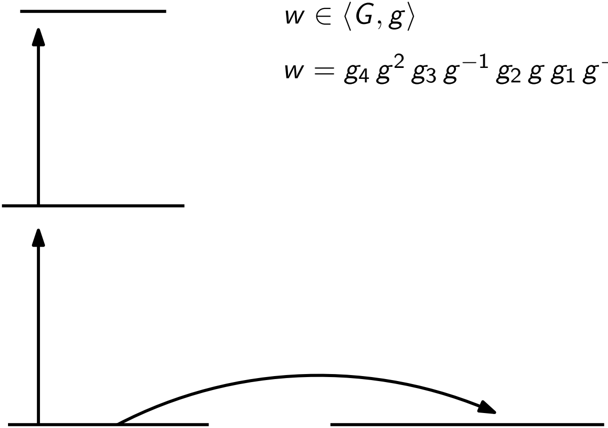


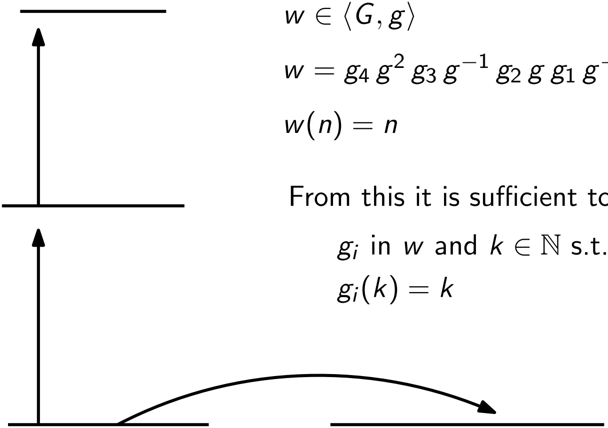




$$w \in \langle G, g \rangle$$

$$w = g_4 \, g^2 \, g_3 \, g^{-1} \, g_2 \, g \, g_1 \, g^{-2} \, g_0$$





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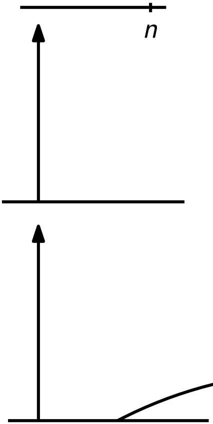
$w(n) = n$

From this it is sufficient to find:

g_i in w and $k \in \mathbb{N}$ s.t.

$g_i(k) = k$



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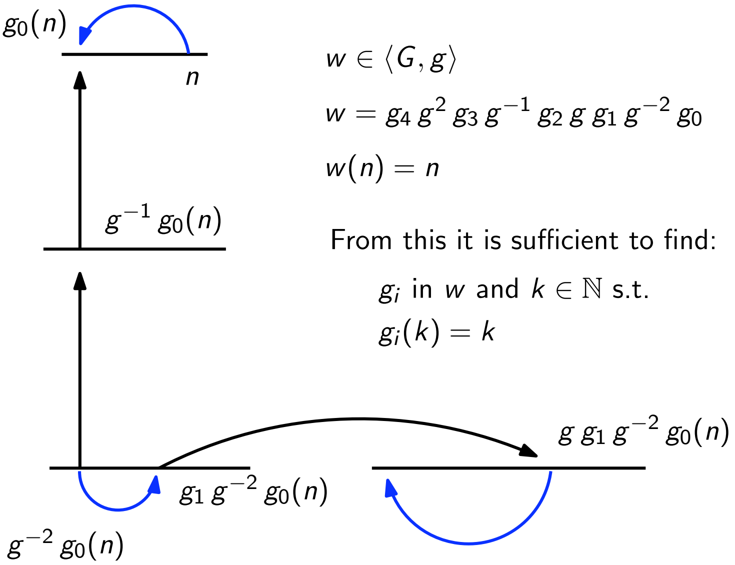

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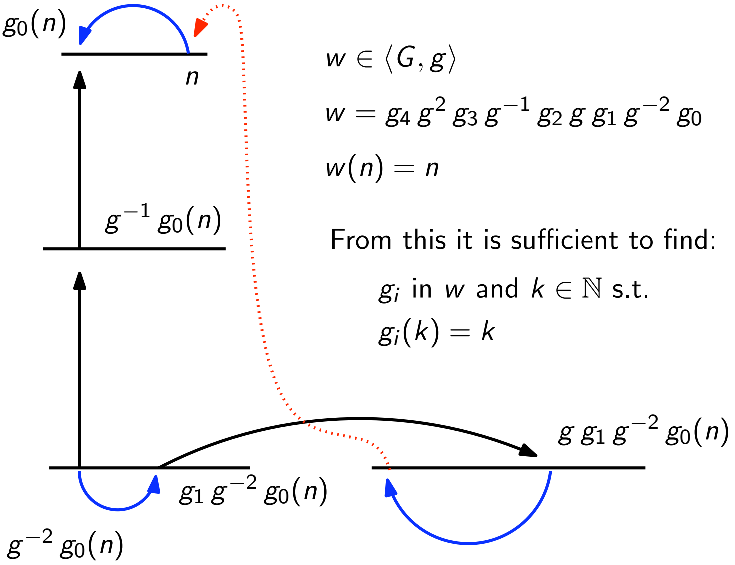

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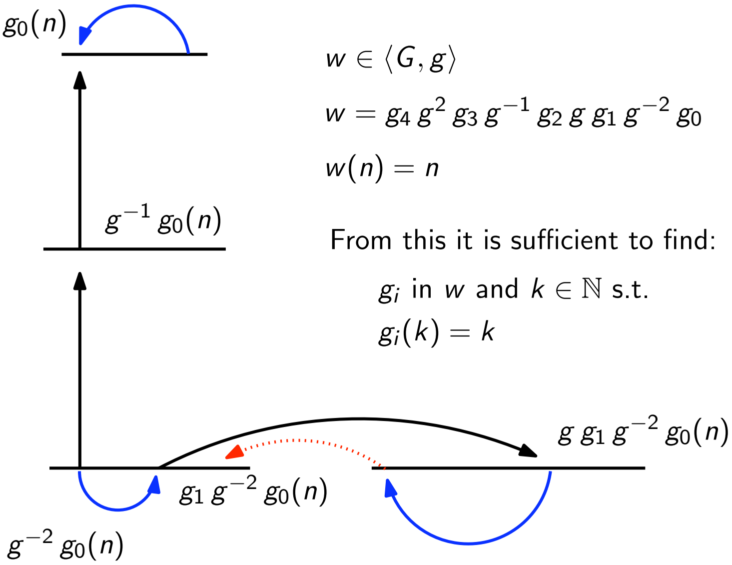

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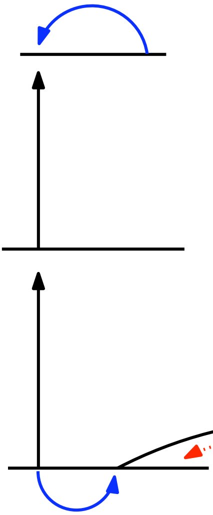

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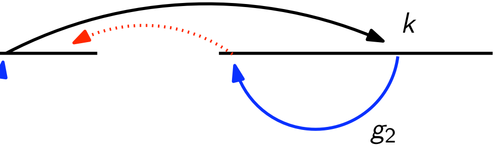

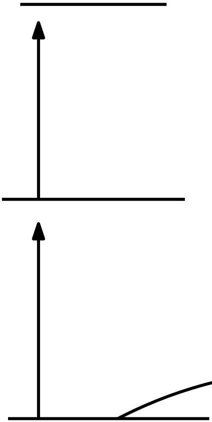

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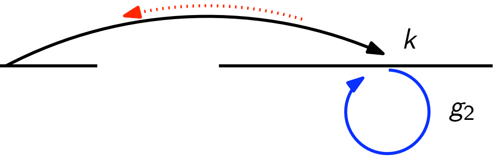


Theorem (Kastermans)
Martin's Axiom implies that there exists a maximal cofinitary group into which any countable group embeds.

Theorem (Blass)
Maximal cofinitary groups cannot be Abelian.

Theorem (Kastermans)
Martin's Axiom implies that there exists a locally finite maximal cofinitary group.

Question: Does there exist a concrete example of a maximal cofinitary group?

Ways of measuring "concreteness":

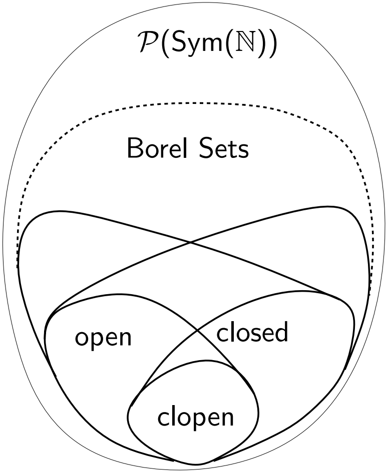
- Topological conditions (closed, compact).
- Borel hierarchy.
- Projective hierarchy.

The topology on $\mathrm{Sym}(\mathbb{N})$ is the product topology:

Basic open sets: $\{f \in \mathrm{Sym}(\mathbb{N}) : f \restriction \{0, \ldots, n\} = s\}$ for $s : \{0, \ldots, n\} \to \mathbb{N}$.

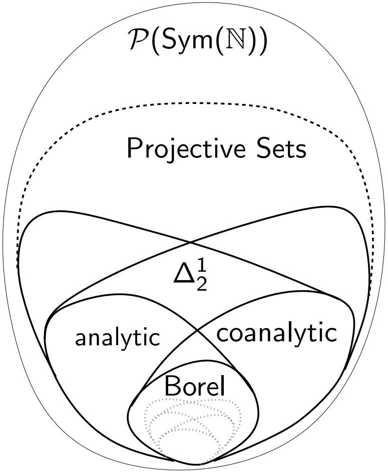
Borel Hierarchy

Generated from the clopen sets by countable unions and complements.



Projective Hierarchy

Generated from the Borel sets by continuous images and complements.



Standard argument: the Axiom of Constructibility gives a maximal cofinitary group of complexity Δ^1_2.

Theorem (Gao and Zhang)

The Axiom of Constructibility implies there exists a maximal cofinitary group with a coanalytic generating set.

Theorem (Kastermans)

The Axiom of Constructibility implies there exists a coanalytic maximal cofinitary group.

thank you

Definition: $\mathfrak{a}_\mathfrak{p}$ the minimal cardinality of a maximal almost disjoint family of permutations.

Definition: $\mathfrak{a}_g$ the minimal cardinality of a maximal cofinitary group.

Must $\mathfrak{a}_\mathfrak{p} = \mathfrak{a}_g$?

Constructing MCG

Grow group step by step $G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle$.

Grow permutation step by step $g_\alpha = \bigcup_{s \in \mathbb{N}} g_{\alpha,s}$.

Description of larger group in terms of smaller:

$$\langle G, g \rangle = (G * F(x))[x := g]$$

$$w(x) \in G * F(x) \qquad w(x) = g_0 x^{k_0} g_1 x^{k_1} \cdots x^{k_m} g_{m+1}$$

Study $w(g_{\alpha,s}) \rightsquigarrow w(g_{\alpha,s+1})$.

good extension: no unavoidable new fixed points.

Gives *free* groups.