Separating Notions of Randomness

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Outline

- Definitions and Basics
- Motivation
- The Result and Idea

Work joint with Steffen Lempp.

The Motivating Question

What is a random real?

Here a real is an infinite binary sequence, an element of $2^{\mathbb{N}}$.

- Null sets (measure theory)
- Bit patterns (Kolmogorov complexity of strings)
- Betting strategies (martingales)

We need to use effective versions of these. We will focus on "effective" martingales.

Definition of Martingale

A *martingale* is a function

$$f : 2^{<\mathbb{N}} \rightarrow \mathbb{R}_0^+$$

such that

$$f(\sigma) = \frac{f(\sigma 1) + f(\sigma 0)}{2}.$$

$f(\sigma)$ is the capital earned after the "betting" on all bits of σ has been completed

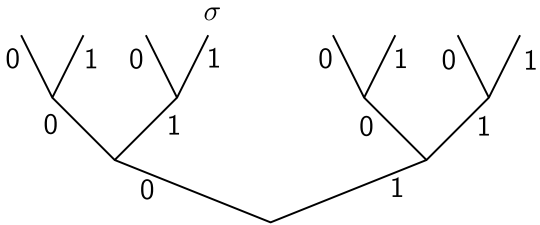
A martingale *succeeds* on a real $A \in 2^{\mathbb{N}}$ iff

$$\limsup_{n \rightarrow \infty} f(A \upharpoonright n) = \infty.$$

Basic Observation I

For every real $A \in 2^{\mathbb{N}}$ there exists a martingale that succeeds on A:

$$E_A(\nu) = \begin{cases} 2^{|\nu|} & \text{if } \nu \prec A \\ 0 & \text{if } \nu \not\prec A \end{cases}$$



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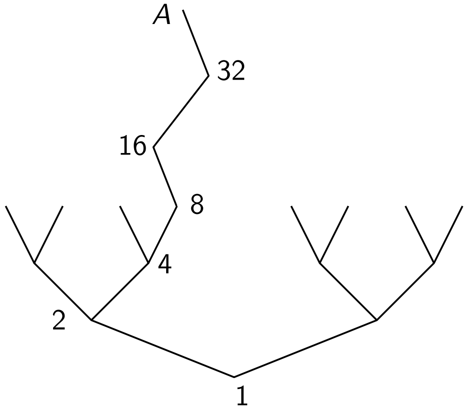
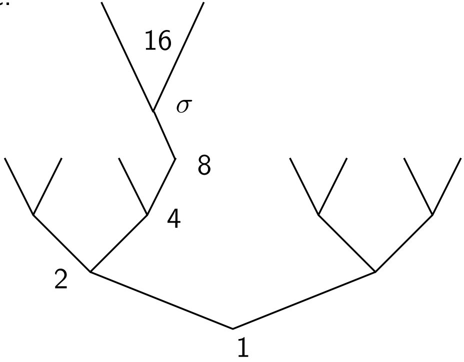


Basic Observation II

Given $\sigma \in 2^{<\mathbb{N}}$ the function E_σ defined by:

$$E_\sigma(\nu) = \begin{cases} 2^{|\nu|} & \text{if } \nu \prec \sigma \\ 2^{|\sigma|} & \text{if } \sigma \prec \nu \\ 0 & \text{otherwise} \end{cases}$$

is a martingale.



Basic Observation III

The set of reals on which a given martingale succeeds is of measure 0.

If

$$S_k = \{A \in 2^{\mathbb{N}} : \exists n \ \left[f(A \upharpoonright n) \geq 2^k \right] \},$$

then

$$\mu(S_k) \leq 2^{-k}.$$

Can "find" such a real by extending initial segments. If we have decided on σ, consider both $f(\sigma 1)$ and $f(\sigma 0)$. One of these is not larger than $f(\sigma)$.

Basic Observation IV

If f and g martingales, then $f + g$ and cf martingales.

With $\langle f_i : i \in \mathbb{N} \rangle$ a sequence of martingales, we can choose constants such that

$$\sum_{i \in \mathbb{N}} c_i f_i$$

is a martingale.

From it we can find a real A on which it does not succeed. Then also none of the f_i succeeds on A.

Notions of Randomness

Definition: *(partial) Computable* martingale:

$f : 2^{<\mathbb{N}} \to \mathbb{Q}_0^+$ (partial) computable.

Definition: *Effective* martingale:

f such that there is a computable function $g : 2^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}_0^+$ increasing in the second coordinate with $f(\sigma) = \lim_{n\to\infty} g(\sigma, n)$.

An effective martingale is effectively approximated from below.

Definition: A real $A \in 2^{\mathbb{N}}$ is computably/partial computably/effectively random iff no computable/partial computable/effective martingale succeeds on A.

Effectively random is called Martin-Löf random.

Well Known Separations

Theorem (Ambos-Spies)

Every partial computable random real is computably random, but not vice versa.

Theorem (Muchnik and Schnorr)

Every Martin-Löf random real is partial computably random, but not vice versa.

Towards Nonmonotonicity

All martingales so far have been monotonic.

Example:

Generate a real by coinflips with the restriction that
if bit 5^i is one, then change bit 3^i to be one also.

Definition of Nonmonotonic Martingale

selection rule: $s : (\mathbb{N} \times \{0,1\})^{<\mathbb{N}} \to \mathbb{N}$, such that $s(\sigma) \notin \mathrm{dom}(\sigma)$.

Example: $s((4,1),(6,0),(2,1)) = 5$

Non-example: $s((4,1),(6,0),(2,1)) = 6$

Definition of Nonmonotonic Martingale

$$\textit{selection rule}: \ s : (\mathbb{N} \times \{0,1\})^{<\mathbb{N}} \to \mathbb{N}.$$

Example: $s((4,1),(6,0),(2,1)) = 5$

$$\textit{stake function}: \ q : (\mathbb{N} \times \{0,1\})^{<\mathbb{N}} \to [0,2].$$

Example: $q((4,1),(6,0),(2,1)) = 1.5$

Assume $d^A(3) = d^A_{\langle s,q\rangle}(3) = 2.5$.
If $A(5) = 1$, then $d^A(4) = 2.5 \cdot 1.5 > 2.5$.
If $A(5) = 0$, then $d^A(4) = 2.5 \cdot (2 - 1.5) < 2.5$.

Definition of Nonmonotonic Martingale

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stake function: $q : (\mathbb{N} \times \{0,1\})^{<\mathbb{N}} \to [0,2]$.

Example: $q((4,1),(6,0),(2,1)) = 1$

Assume $d^A(3) = d^A_{\langle s,q \rangle}(3) = 2.5$.
If $A(5) = 1$, then $d^A(4) = 2.5 \cdot 1 = 2.5$.
If $A(5) = 0$, then $d^A(4) = 2.5 \cdot (2 - 1) = 2.5$.

Definition of Nonmonotonic Martingale

selection rule: $s : (\mathbb{N} \times \{0,1\})^{<\mathbb{N}} \to \mathbb{N}$.

Example: $s((4,1),(6,0),(2,1)) = 5$

stake function: $q : (\mathbb{N} \times \{0,1\})^{<\mathbb{N}} \to [0,2]$.

Example: $q((4,1),(6,0),(2,1)) = 0.5$

Assume $d^A(3) = d^A_{\langle s,q \rangle}(3) = 2.5$.
If $A(5) = 1$, then $d^A(4) = 2.5 \cdot 0.5 < 2.5$.
If $A(5) = 0$, then $d^A(4) = 2.5 \cdot (2 - 0.5) > 2.5$.

The Question

Definition: A real is Kolmogorov-Loveland random if no nonmonotonic martingale with both s (the selection rule) and q (the stake function) partial computable wins.

Any Martin-Löf random real is Kolmogorov-Loveland random, and any Kolmogorov-Loveland random real is partial computably random.

Question: (Muchnik 1998, Ambos-Spies and Kučera 2000)
Do the notions of Martin-Löf randomness and Kolmogorov-Loveland randomness coincide?

Restricting Kolmogorov-Loveland martingales

Restrictions on s (J. Miller and Nies) — determine s from some function h:

$$s(\tau) = h(|\tau|).$$

Definition: A real is *permutation random*, is no nonmonotonic martingale with s obtained from a computable permutation h succeeds on it.
Definition: A real is *injective random* if no nonmonotonic martingale with s obtained from a computable (injection) h succeeds on it.

Theorem (Muchnik)

Any Kolmogorov-Loveland random real is partial computably random, but not vice versa.

The proof actually shows the separation for permutation randomness.

Our Theorem

Theorem (Kastermans and Lempp)

Not every injective random real is Martin-Löf random.

Outline of the Proof

Theorem (Kastermans and Lempp)

Not every permutation random real is Martin-Löf random.

Strategy:

"Find" real A and effective martingale M such that A is permutation random but M succeeds on A.

"Find" a single martingale that ensures the permutation randomness.

Defeat it.

Two Observations

If $\langle f_i : k \in \mathbb{N} \rangle$ are monotonic martingales, can choose constants c_i such that

$$\sum_{i \in \mathbb{N}} c_i f_i$$

is a martingale.

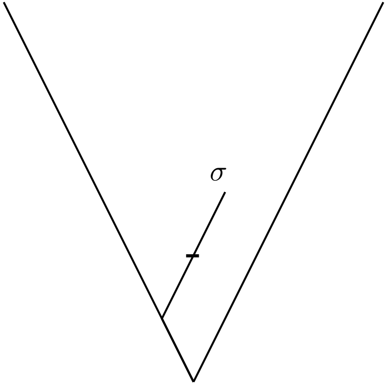
Can get the effective martingale by summing different E_σ with appropriate scaling factors.

$$1 = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Add a new E_σ if some strategy succeeds; if the σ become long enough the capital grows without bound.

Monotonize

$$d^{\text{expec}}_{\langle h,q \rangle}(\sigma) :=$$



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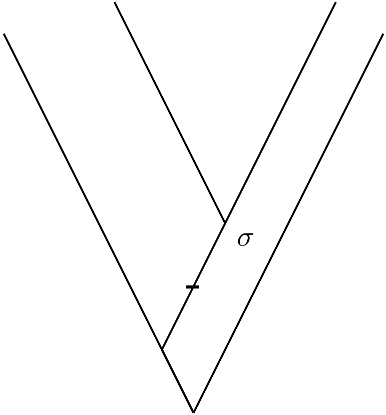

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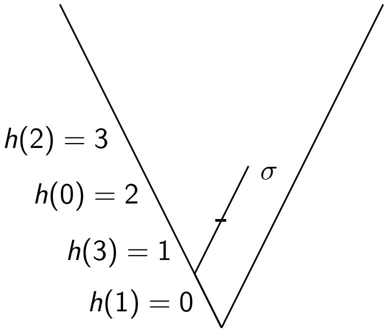

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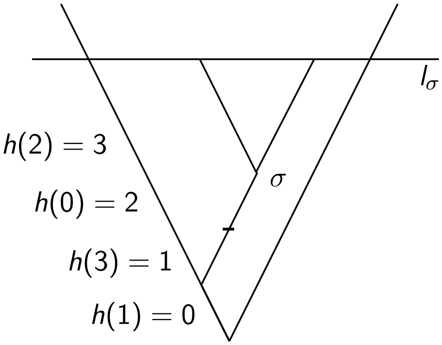
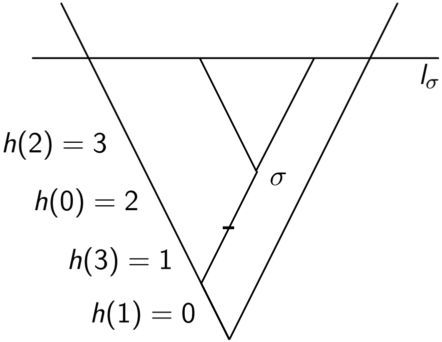


Monotonize

$$d^{\text{expec}}_{\langle h,q\rangle}(\sigma) := \frac{1}{2^{(l_\sigma - |\sigma|)}} \sum_{\substack{p \in 2^{l_\sigma} \\ \sigma \prec \tau}} d^{\tau}_{\langle h,q\rangle}(n_\sigma)$$



Partiality

$\langle (h_i, q_i) : i \in \mathbb{N} \rangle$ enumeration of all partial permutation martingales.

Sum of earlier permutation martingales related to the new one.

New one diverges where sum is small, ... or not.

f total monotonic, (h_i, q_i) next partial permutation martingale to look at.

Make total by betting even.

Thank you.

Slides and paper available at http://www.bartk.nl/

Injective Randomness

Consider $\{(h_0, q_0), \ldots, (h_n, q_n)\}$ and $\sigma : \omega \rightharpoonup 2$ finite. When can we tell when the betting on σ is done?

For most σ we have no hope, but with a guess we can for certain σ.

The type of a bit k is the set of i such that $k \in \mathrm{ran}(h_i)$.

$$\mathrm{Av}_T(\sigma) = \sum_{\tau \in 2^l, \sigma \prec \tau} 2^{-(l-|\sigma|)} (\sum_{j \in T} f_j^{\tau}(t))$$

where t and l are sufficiently large.