Fractal Techniques in Topological Semantics for Modal Languages

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(Joint work with Tamar Lando, UC Berkeley)

1. Two Bibliographies
2. Modal Language, Kripke Structures, and Trees
3. Topological Semantics for Modal Logic
4. Koch Curve and Topological Completeness
5. Sierpinski Carpet and Menger Cube

Classic Work in Topo ML, Fractal Geometry, Tree Topology

S. Kripke. *Semantical analysis of modal logic i*, Zeitschr. f. math. Logik und Grund. d. Mathematik **9** 67–96, (1963).

J. C. C. McKinsey and A. Tarski, *The algebra of topology*, Ann. of Math. (2) **45**, 141–191, (1944).

P. Blackburn, M. de Rijke, and Y. Venema. *Modal logic.* Cambridge tracts in theoretical computer science, Vol. 53. CUP, Cambridge, (2001).

M. Barnsley, *Fractals Everywhere,* Academic Press, (1988).

B. Mandelbrot, *The Fractal Geometry of Nature,* W. H. Freeman & Co., (1982).

Schroeder, M. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise*, New York: W. H. Freeman, pp. 41-45, (1991).

D. Scott. *Domains for Denotational Semantics*. International Colloquium on Automata, Languages and Programming (ICALP), Springer LNCS, Vol. 140, 577–613, (1982).

Topological Revival in ML: 1997–Present, Partial Bibliography I I

- S.N. Artemov, J.M. Davoren, and A. Nerode, *Modal logics and topological semantics for hybrid systems.* Mathematical scineces institute, Technical Report 97-05, (1997).
- M. Aiello, J. van Benthem, and G. Bezhanishvili, *Reasoning about space: the modal way*, Journal of Logic and Computation **13**6, 889–920, (2003).
- J. van Benthem, G. Bezhanishvili, B. ten Cate, and D. Sarenac, *Modal Logics for Products of Topologies. Studia Logica* (2006).
- Guram Bezhanishvili, Mai Gehrke, *Completeness of S4 with respect to the real line: revisited*, Ann. Pure Appl. Logic 131(1-3): 287-301 (2005)
- P. Kremer and G. Mints (2007), "Dynamic Topological Logic," in *Handbook of Spatial Logic*, M. Aiello, I. Pratt-Hartmann, J. van Benthem (Eds.), Springer, 565-606.
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- G. Mints, *A completeness proof for propositional S4 in cantor space*, Ch. 6, In *Logic at Work*, Kluwer, Publishing, (1998)

Fractals and $\mathbb{Q} \times \mathbb{Q}$

Making a fractal copy of $\mathbb{Q} \times \mathbb{Q}$

1

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0

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11

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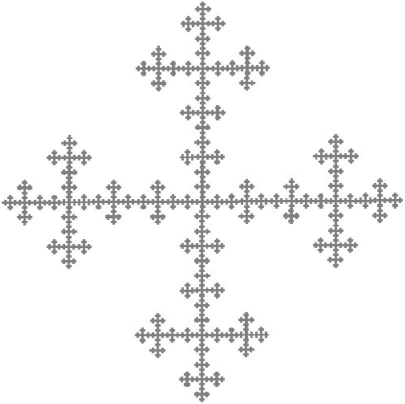
Viscek Fractal



The origin of the work on fractals

A product construction in TML over the rationals.

Language, Frames, Models, Truth in a Model

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- Notice similarity with Kuratowski's Topological Axioms

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- Extensivity: $I(A) \subseteq A$ (T)

Kripke and Transitive, Reflexive, Rooted Relational Models

Theorem ($S4$ Completeness)

The modal Logic S4 is sound and complete for:

- *the class of all frames with transitive, reflexive R*

Proof.

for proofs, see for example [3] or [1] □

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- *the class of all finite transitive, reflexive frames*
- *the class of all rooted, finite, transitive, reflexive frames*

Proof.

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T_2 as Modal Frame

Definition (Sequences of 0s and 1s; Immediate Successors; Tree Frames)

- Let $\Sigma = \{0, 1\}$ and Σ^* be the set of all finite strings in the language Σ including $\langle \cdot \rangle$, the empty string. Let Σ^o be the set of all countably infinite strings over Σ, and let $\Sigma^+ = \Sigma^* \cup \Sigma^o$.

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- We let $s_i : \Sigma^* \to \Sigma^*$ for $i \in \{0, 1\}$ be the function defined by $s_i(x) = x * i$. Thus for example $s_0(1) = 10$, and $s_1(110) = 1101$. We call $s_0(x)$ the left successor of x and $s_1(x)$ the right successor of x.

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- Define R_2xy for $x, y \in \Sigma^*$ as the transitive reflexive closure of $s_0 \cup s_1$.

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- $T_2 = (\Sigma^*, R_2, \langle\cdot\rangle)$

Lemma (Unraveling)

For any finite, transitive, reflexive, rooted, model $\mathcal{M}$ with a root x, there is a valuation V on T_2 such that for any modal formula ϕ,

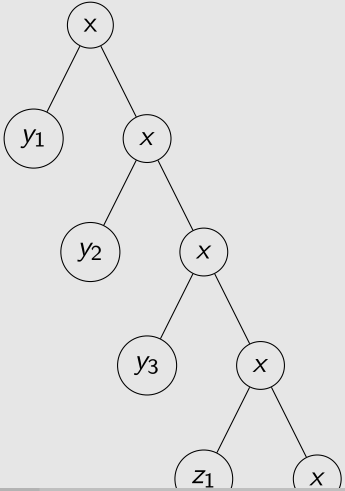
$$\mathcal{M}, x \models \phi \Leftrightarrow \mathcal{T}_2, \langle \cdot \rangle \models \phi$$

Completeness of $S4$ for T_2

It then follows from Lemma 3 that every nontheorem of $S4$ can be shown false on some model based on the frame T_2. If ϕ is not a theorem of $S4$, then by Theorem 1 there is some finite rooted frame $\mathcal{F} = (U, R, x)$ with a valuation V such that $(\mathcal{F}, V), x \models \neg\phi$, but then by Claim 3, there is a valuation V' such that $(T_2, V'), \langle \cdot \rangle \models \neg\phi$. Thus any nontheorem fails on T_2 and the singleton T_2 is complete for $S4$.

Finite Modal Frames are Readily Unraveled to T_2

Finite Frame



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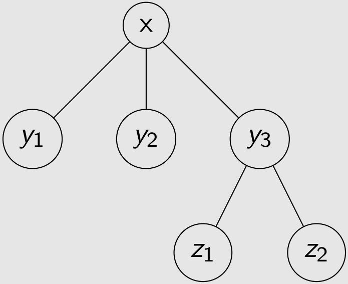
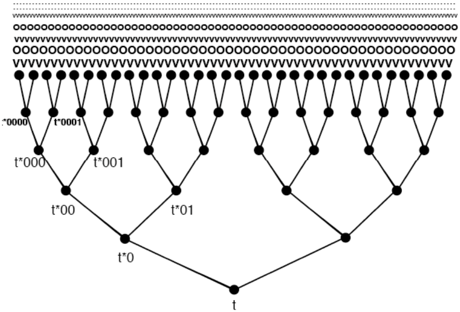


Finite Tree Unravelled to Full Binary Tree



Topological Models, Truth in a Model

Topological Semantic, Truth in a Model

- Topology $\mathcal{X} = (X, \mathcal{O})$:
  (i) $X, \emptyset \in \mathcal{O}$,
  (ii) $A, B \in \mathcal{O} \Rightarrow A \cap B \in \mathcal{O}$,
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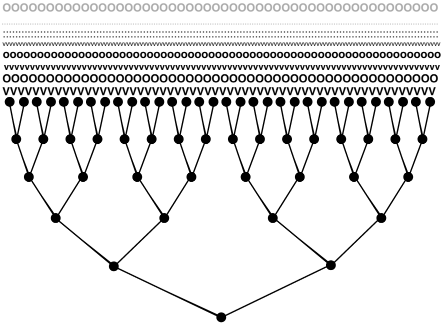
9. *Wilson Tree (or The Full Binary Tree with Limits),* $\mathbb{T}_2^+$*, equipped with the topology generated by the finite initial segments. (This paper)*

Wilson tree $\mathbb{T}_2^+$ is a Limit of the Standard Infinite Binary Tree



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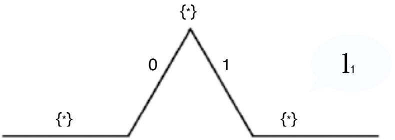
- We build a function $l : [0,1] \to \mathbb{T}_2^+$ directly,
- but, it is rather obvious from l what h and g are, that is, the fractal inspiration behind the proof is kept transparent.

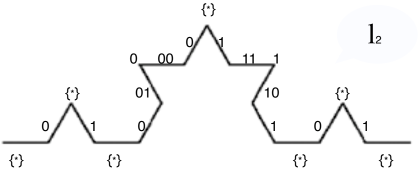
Function g: From Koch Curve to $[0, 1]$

$$[0, \tfrac{1}{4}], (\tfrac{1}{4}, \tfrac{1}{2}), [\tfrac{1}{2}], (\tfrac{1}{2}, \tfrac{3}{4}), [\tfrac{3}{4}, 1]$$

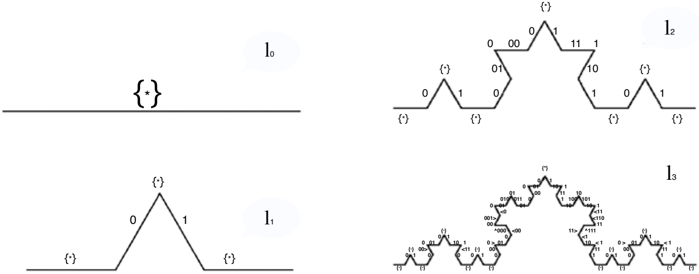


$$[0, \tfrac{1}{16}], (\tfrac{1}{16}, \tfrac{1}{8}), [\tfrac{1}{8}], (\tfrac{1}{8}, \tfrac{3}{16}), [\tfrac{3}{16}, \tfrac{1}{4}], (\tfrac{1}{4}, \tfrac{5}{16}], (\tfrac{5}{16}, \tfrac{3}{8}), [\tfrac{3}{8}], (\tfrac{3}{8}, \tfrac{7}{16}), [\tfrac{7}{16}, \tfrac{1}{2}), [$$

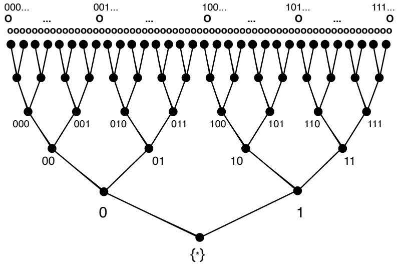
$$(\tfrac{1}{2}, \tfrac{9}{16}], (\tfrac{9}{16}, \tfrac{5}{8}), [\tfrac{5}{8}], (\tfrac{5}{8}, \tfrac{11}{16}), [\tfrac{11}{16}, \tfrac{3}{4}), [\tfrac{3}{4}, \tfrac{13}{16}], (\tfrac{13}{16}, \tfrac{7}{8}), [\tfrac{7}{8}], (\tfrac{7}{8}, \tfrac{15}{16}), [\tfrac{15}{16}, 1]$$

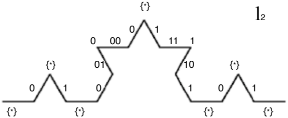


Function *h*: Labeling of Koch Curve with the nodes of $\mathbb{T}_2^+$

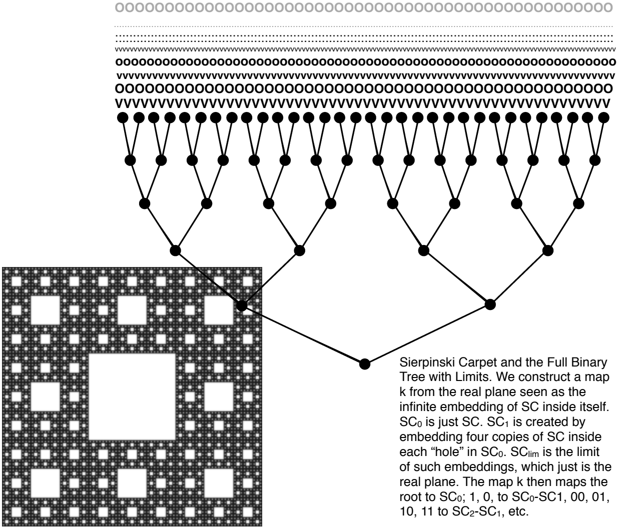


Function g: Koch Curve to Tree $\mathbb{T}_2^+$



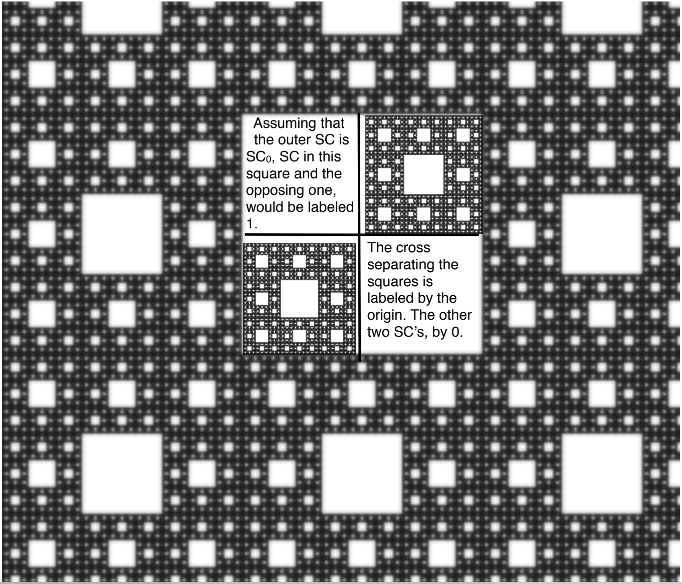


Function K: From Sierpinski Carpet to $[0,1]^2$



Sierpinski Carpet and the Full Binary Tree with Limits. We construct a map k from the real plane seen as the infinite embedding of SC inside itself. SC_0 is just SC. SC_1 is created by embedding four copies of SC inside each “hole” in SC_0. SC_{lim} is the limit of such embeddings, which just is the real plane. The map k then maps the root to SC_0; 1, 0, to SC_0-SC1, 00, 01, 10, 11 to SC_2-SC_1, etc.

Sierpinski Embedding



Function j: From Menger Sponge to $[0, 1]^3$