GROUPS OF FINITE MORLEY RANK AND RECOGNIZING PSL_2

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ABSTRACT. The study of groups of finite Morley rank generalizes the study of algebraic groups over algebraically closed fields with Morley rank generalizing the usual dimension function. In fact, these two classes of groups are suspected to have a deep connection as witnessed by the Cherlin-Zil'ber conjecture: an infinite simple group of finite Morley rank is isomorphic (as an abstract group) to an algebraic group over an algebraically closed field.

The goal of this talk is to give a brief introduction to groups of finite Morley rank and the Cherlin-Zil'ber conjecture. Ultimately, the talk will focus on recent work on how to recognize the "smallest" of the infinite simple algebraic groups: PSL_2.

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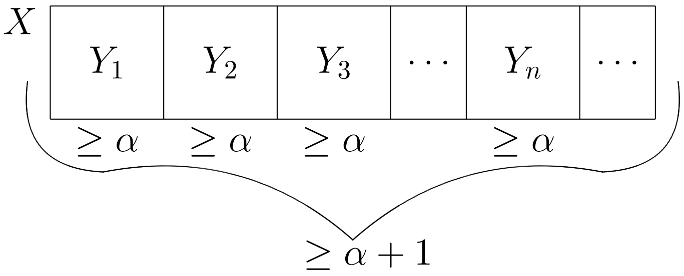
1. Morley Rank (simplified)

Defining Morley Rank

- Pick a language $\mathcal{L}$.
- Pick an ($\aleph_0$-saturated) $\mathcal{L}$-structure $\mathbb{M}$.
- Pick an $\mathcal{L}(M)$-definable subset of M^n, call it X.
- RM($-$) assigns to X a "dimension" that is -1, an ordinal, or ∞ as follows:
  - RM(X) $= -1$ iff X is empty.
  - RM(X) $\geq \alpha + 1$ iff $\exists$ infinitely many definable subsets $Y_1, Y_2, Y_3, \ldots$ such that the Y_i are disjoint and each have Morley rank at least α:



  - RM(X) $\geq \lambda$ for λ a limit ordinal iff RM(X) $\geq \alpha$ for all $\alpha < \lambda$
  - RM(X) $= \infty$ iff RM(X) $> \alpha$ for all ordinals α
- RM($\mathbb{M}$) := RM(M)

Remark.

- Note that RM(X) $= 0$ iff X is finite.

Example. Let $\mathbb{K}$ be an $\langle\langle$ saturated $\rangle\rangle$ algebraically closed field considered in the language $\mathcal{L} = \{+, -, \cdot, 0, 1\}$.

(1) What is $\mathrm{RM}(\mathbb{K})$?

- The $\mathcal{L}(K)$-definable subsets of K are the finite sets, the cofinite sets, $\emptyset$, and K. $\langle\langle$ use q.e. $\rangle\rangle$
- $\mathrm{RM}(K) = 1$, so $\mathbb{K}$ has finite Morley rank.

(2) In general, a definable subset X of K^n has Morley rank equal to its usual Zariski dimension.

$\langle\langle$ Zariski dimension of an irreducible X is the transcendence degree of $\mathrm{Frac}(K[t_1, \ldots, t_n]/I(X))$ over K. $\rangle\rangle$

(3) [**OP**] The subset G of K^5 given by $\{(a, b, c, d, e) : (ad - bc)e = 1\}$ can be identified with $\mathrm{GL}_2(\mathbb{K})$ by

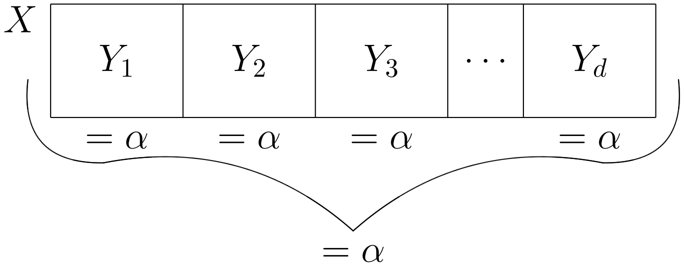
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \left(a, b, c, d, (ad - bc)^{-1}\right).$$

- Group operations $\{*, {}^{-1}, e\}$ can be defined on G using $\mathcal{L}(K)$.
- The resulting group carries a Morley rank on its $\mathcal{L}(K)$-definable sets.
- This group is an example of an affine algebraic group.
  - The underlying set is the set of common zeros of polynomials in $K[t_1, \ldots, t_n]$ ($n = 5$ above).
  - The group operations are definable by polynomials.

Definition. A "group of finite Morley rank" is an expansion of a group $(G; \cdot, {}^{-1}, 1, \ldots)$ such that G has finite Morley rank (FMR).

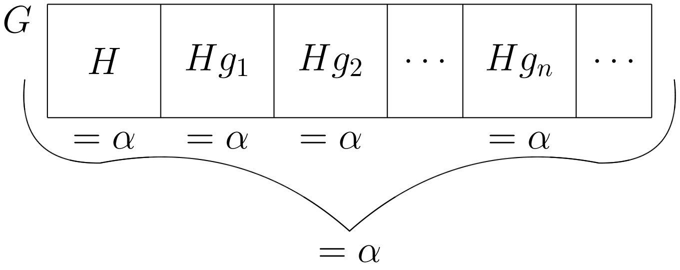
Morley Degree

- Let $\mathbb{M}$ and X be as before such that $\mathrm{RM}(X) = \alpha$ is an ordinal.
- X has Morley degree d iff d is the maximum such that



Example. Let G be a group with ordinal Morley rank α, and H a proper definable subgroup.

(1) If $\mathrm{RM}(H) = \alpha$ each coset of H has rank α as well.



  - H must have finite index in G.
  - The Morley degree of H is less than the Morley degree of G.
  - If H has Morley degree 1, then the Morley degree of G is the index of H in G.

(2) DCC on definable subgroups: rank or degree must go down.

Properties of groups of finite Morley rank

- DCC on definable subgroups.
  - Gives a nice inductive setting.
  - Yields a definable closure for non-definable subgroups, which retains many properties of the original subgroup.
- (Macintyre) An abelian GFMR is "definable divisible × bounded exponent".

  $\langle\langle\ G \cong \left(\bigoplus_p (\bigoplus_{I_p} \mathbb{Z}(p^\infty))\right) \oplus (\bigoplus_I \mathbb{Q}) \oplus B$ where B has bounded exponent. $\rangle\rangle$
- (Cherlin-Macintyre) An infinite division ring has FMR iff it is an algebraically closed field.
- Structures interpretable in structures of FMR have FMR.

Example.

(1) Affine algebraic groups over an algebraically closed field, $\mathbb{K}$, have FMR: $\mathrm{GL}_n(\mathbb{K})$, $\mathrm{PSL}_n(\mathbb{K})$, finite groups, ...

(2) $\mathbb{Z}$ does NOT have FMR because $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq \cdots$

Cherlin-Zil'ber Conjecture

Conjecture (Cherlin-Zil'ber Conjecture)**.** *An infinite simple group of finite Morley rank is isomorphic to an (affine) algebraic group over an algebraically closed field.*

- By conjugacy and structure theorems for the Sylow 2-subgroups, the analysis of a simple GFMR is divided as follows:

  even: S contains an ∞ elementary abelian 2-group and does NOT contain a $\mathbb{Z}(2^\infty)$.

  odd: S contains NO ∞ elementary abelian 2-group and does contain a $\mathbb{Z}(2^\infty)$.

  mixed: S contains BOTH an ∞ elementary abelian 2-group and a $\mathbb{Z}(2^\infty)$.

  degenerate: $S = 1$.
- Having an ∞ elementary abelian 2-group corresponds to having a field of char. 2 around ($\mathbb{K}^+$).
- Having a $\mathbb{Z}(2^\infty)$ corresponds to having a field of char. not 2 around ($\mathbb{K}^\times$).
- If the conjecture is true, mixed and degenerate type infinite simple GsFMR should not exist.
- A recent theorem of Alınel, Borovik, and Cherlin eliminates mixed type for infinite simple GsFMR and identifies those in even type as algebraic (so we are half done).
- Eliminating degenerate type amounts to a Feit-Thompson Theorem and has been referred to by some as "the wild west of the subject".

Recognizing PSL_2

Let G be a simple GFMR. What conditions on G can we look for to recognize G as being isomorphic to some $\mathrm{PSL}_2(\mathbb{K})$ with $\mathbb{K}$ algebraically closed?

Structurally: In $\mathrm{PSL}_2(\mathbb{K})$ we have the following:

- the Sylow 2-subgroups are elementary abelian in characteristic 2,
- the Sylow 2-subgroups are isomorphic to $\mathbb{Z}(2^\infty)$ in characteristic not 2, and
- more along these lines.

Via an Action: With $\mathrm{PSL}_2(\mathbb{K})$ acting naturally on the projective line we have the following:

- the action is sharply 3-transitive, or
- the action is 2-transitve and a point stabilizer is isomorphic to $\mathbb{K}^+ \rtimes \mathbb{K}^\times$.

Theorem (Nesin - 1990)**.** *An infinite sharply 3-transitive GFMR is isomorphic to* $\mathrm{PSL}_2(\mathbb{K})$ *for* $\mathbb{K}$ *an algebraically closed field.*

Conjecture. *If G is an infinite simple GFMR acting 2-transitively on a set such that a point stabilizer G_x splits as $U_x \rtimes G_{xy}$ with U_x abelian, then G is isomorphic to* $\mathrm{PSL}_2(\mathbb{K})$ *for $\mathbb{K}$ an algebraically closed field.*

$\langle\langle$ *The simplicity hypothesis simply simplifies the situation.* $\rangle\rangle$

Progress on the conjecture

Setup.

- G is an infinite simple GFMR acting 2-transitively on a set X. Fix $\infty \neq 0 \in X$.
- $G_\infty = U_\infty \rtimes G_{0,\infty}$ with U_∞ abelian.
- Set $U := X \setminus \{\infty\}$ (our intuition is that $X = P(\mathbb{K})$ and $U = \mathbb{K}^+$).

Fact.

(1) U_∞ acts regularly on U, placing a group structure on U such that $U \cong U_\infty$.

(2) $G_{0,\infty} \hookrightarrow \mathrm{Aut}(U)$.

(3) U is a vector space over some $\mathbb{F}_p$ or $\mathbb{Q}$.

Theorem (De Medts - Tent)**.** *If U has characteristic not 2 and if $\mathbb{K} := \mathrm{End}_{G_{0,\infty}}(U)$ is infinite, then $G \cong \mathrm{PSL}_2(\mathbb{K})$ with $\mathbb{K}$ an algebraically closed field.*

Theorem (W)**.** *Same as last theorem, but in characteristic 2.*

Remark. These theorems combine to prove the conjecture under the additional (and seemingly strong) hypothesis that $\mathrm{End}_{G_{0,\infty}}(U)$ is infinite. This is however automatically satisfied in characteristic 0.

Outline of the proofs

(1) Show $G_{0,\infty}$ acts irreducibly on U. (Follows from earlier work in characteristic not 2.)

(2) Conclude that $\mathrm{End}_{G_{0,\infty}}(U)$ is an algebraically closed field. ⟨⟨ Schur's lemma ⟩⟩

(3) Use RM to work in a minimal counter-example appealing to linear algebra.

(4) Prove that G behaves a lot like $\mathrm{PSL}_2(K)$.

(5) Arrive at a contradiction.

Proof of (1) in characteristic 2

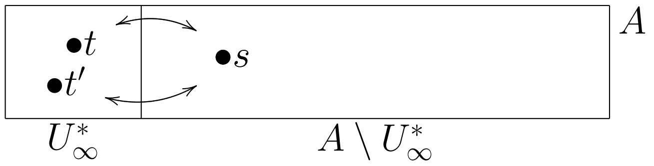
(i) Rephrase the problem:

- We actually show that $G_{0,\infty}$ acts transitively on U^*.
- This is equivalent to $G_{0,\infty}$ acting transitively on U^*_∞ (by conjugation).
- It suffices to show that $A := \bigcup_{x \in X} U^*_x$ is a G-conjugacy class:

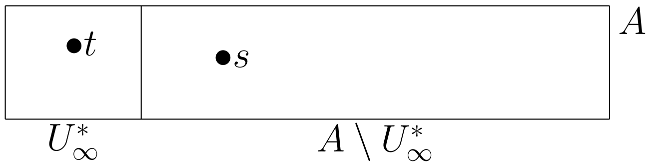
$$x, y \in U^*_\infty \text{ and } x^g = y \implies g \in G_\infty \implies g = uh \text{ with } u \in U_\infty, h \in G_{0,\infty}$$

$$\implies y = x^g = x^{uh} = x^h.$$

- Suffices to show that for all $t \in U^*_\infty$, $s \in A \setminus U^*_\infty$ we have $t \sim s$



(ii) Fix $t \in U_\infty^*$ and $s \in A \setminus U_\infty^*$.



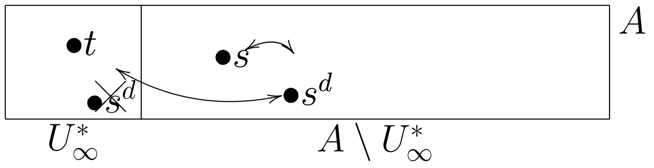
- A consists of involutions ($\mathrm{char}(U) = 2$ and $U_\infty \cong U$).
- $\langle s, t\rangle = \langle st\rangle \rtimes \langle s\rangle$ is a dihedral group.

(iii) $|st|$ is odd: $t \sim s$ and we are done (think $ststst = 1 \implies (st)s(ts) = t$).

(iv) $|st|$ is even: cannot happen; it forces $s \in U_\infty^*$.

(v) $|st| = \infty$: Let $d(\langle st\rangle)$ be the definable closure of $\langle st\rangle$.

- $d(\langle st\rangle)$ is abelian and inverted by s and t, since $\langle st\rangle$ has these properties.
- $d(\langle st\rangle) = D \times C$ for D divisible and C finite cyclic (using the theory of GsFMR).
- Write $st = d^2c$.
- Then, $s^d = d^{-1}sdss = d^{-2}s = cc^{-1}d^{-2}s = ct$, so $s^dt = c$ has finite order.
- $c = 1$: $s^d = t$, and we are done.
- $c > 1$:



  - If $s^d \in A \setminus U_\infty^*$, then above implies $s^d \sim t$.
  - s^d can not be in U_∞^* (it forces $s \in U_\infty^*$).

Summary

Theorem. *If G be an infinite simple GFMR acting 2-transitively on a set such that a point stabilizer G_x splits as $G_x = U_x \rtimes G_{xy}$ with U_x abelian, then G is isomorphic to* $\mathrm{PSL}_2(\mathbb{K})$ *for* $\mathbb{K}$ *an algebraically closed field provided that* $\mathrm{End}_{G_{0,\infty}}(U)$ *is infinite.*

Questions.

(1) Must $\mathrm{End}_{G_{0,\infty}}(U)$ necessarily be infinite?

(2) Can we relax the condition that U_x is abelian to nilpotent or solvable or...?

THANK YOU!